SOME GRAPHICAL PROBLEMS RELATED TO SCORE SEQUENCES, PARTIAL ORDERS AND FINITE TOPOLOGIES

A thesis submitted for the degree of
Doctor of Philosophy at the
University of Keele

by

PETER AVERY

Department of Mathematics,
University of Keele,
September 1978.

TO MY PARENTS

P R E F A C E

This thesis presents the results of research carried out by the author at the University of Keele, 1972 - 75. Except where acknowledged otherwise the work reported here is claimed as original, and has not previously been submitted for a higher degree of this or any other university.

I wish to thank

Mr. Keith Walker, my supervisor, for his wonderful guidance and help throughout,

the Science Research Council for the financial support of a Research Studentship during the period 1972 - 75,

my parents for their unfailing encouragement and support during my studies at school and university,

and Mrs. Lorraine Jarvis for her excellent typing of the thesis.

A B S T R A C T

This thesis presents a study of some topics and problems of graph theory that are related to partial orders. It is an attempt to develop and utilise the links between score sequences of oriented graphs, different types of partial orders, and combinatorial aspects of finite topologies.

In the first chapter Landau's concept of the score structure of a tournament is extended to that of the score sequence of an oriented graph, and the condition is given for an arbitrary integer sequence to be the score sequence of some oriented graph. The proof involves the construction of a specific oriented graph $\tilde{D}(S)$ with the given score sequence S. Landau's condition for the score structure of a tournament is deduced as a simple consequence by constructing a tournament $\tilde{T}(L)$ with given score structure L. It is shown that $\tilde{D}(S)$ is a partial order for every S, and an enumeration of score sequences /labelled score sequences provides a lower bound for the number of partial orders / T_0 topologies. The partial order $\tilde{D}(S)$ has the minimum number of arcs among the oriented graphs with score sequence S, and $\tilde{T}(L)$ has the maximum number of upsets among the tournaments with score structure L. We also establish that given two oriented graphs with the same score sequence, one can be transformed to the other by successively transforming appropriate intransitive triples to transitive triples or vice versa. The proof of an analogous earlier result of Ryser for tournaments reduces to a particular case of the proof of this result.

The second chapter begins with an introduction to some of the ideas of paired comparison experiments. If indifference is not permitted then the preference pattern induced by such an experiment is a tournament, and a consistent preference pattern is a linear order. If indifference is permitted then the induced preference pattern is an oriented graph, and one interpretation of consistency is that indifference as well as preference should be transitive, in which case consistent preference patterns are weak orders. We show that those oriented graphs which are specified up to isomorphism by their score sequences are weak orders, and establish some further results relating weak orders and score sequences. In a similar vein, the condition is given for a tournament to be specified up to isomorphism by its score structure.

The third chapter is about semiorders, which, like weak orders, are special types of partial orders. Luce and others have suggested that preferences in paired comparison experiments become recognisable only when of sufficient magnitude. Under this assumption, consistent preference patterns are characterised as semiorders. The proof relies upon reformulating the definition of a semiorder in terms of its score sequence. There is an interesting application of this result to significance testing in statistics. The number of semiorders with n vertices is shown to be the nth Catalan number and labelled semiorders are counted. Harary's reconstruction conjecture is proved for semiorders. Then we consider extensions of the Kendall and Babington Smith definition of the coefficient of consistency of a paired comparison experiment to those experiments in which indifference is permitted.

In the final chapter, the links between finite topologies and transitive digraphs are studied. It is well known that there is a natural one-to-one correspondence between the finite topologies and transitive digraphs with the same point (vertex) set. This correspondence is exploited to derive the graphical analogues of topological properties such as connectivity, maximal connectivity, fineness, the T_0 separation axiom and closure. Certain binary operations between topologies which produce other topologies, such as union, intersection and the cartesian product, are also shown to have simple analogues in graph theory. The number of n-point maximal connected topologies is counted and related to the counting series for rooted trees. Then the following problem is investigated: given n, for which values of r is there an n-point topology with r open sets? The cardinalities of topologies are studied by considering their associated digraphs, and it transpires that it is necessary to consider only those transitive digraphs which are partial orders. Various results are proved, including existence and non-existence criteria, enabling a complete solution to be given for $n \leqslant 9$. In conclusion, three conjectures about cardinalities are put forward.

CONTENTS

T E R M I N O L O G Y

A N D   N O T A T I O N

Standard terminology and notation are used wherever possible. Definitions are usually given as they are required and mostly conform with those of [17]. The completion or absence of a proof is indicated by the symbol //.

CHAPTER 1. SCORE SEQUENCES OF ORIENTED GRAPHS

SECTION 1.1 Preliminary definitions and introduction

Binary relations

A *binary relation* R is a finite set V of vertices together with a set of distinct ordered pairs of vertices. If the ordered pair uv is in the relation, we say uRv.

A relation is *reflexive* if vRv for all $v \in V$; it is *irreflexive* if vRv for no $v \in V$.

A relation is *symmetric* if uRv implies vRu; it is *asymmetric* if uRv precludes vRu.

A relation is *transitive* if for distinct vertices u, v and w, uRv and vRw imply uRw.

A relation is *complete* if for distinct u and v, either uRv or vRu.

An *equivalence relation* is reflexive, symmetric and transitive.

Labelled digraphs

A *labelled digraph (directed graph)* D = (V, R) of order n is a finite vertex set V of cardinality n together with an irreflexive binary relation R defined on V.

Adjacency and incidence

Vertex u is *adjacent to (dominates)* vertex v if uRv, in which case the *arc* uv is in D and v is *adjacent from (dominated by)* u. The arc uv is *incident* with u and v.

A symmetric indifference relation I is defined on V by uIv if neither uRv nor vRu.

Isomorphism

Two labelled digraphs are *isomorphic* if there is a one-to-one correspondence between their vertex sets which preserves their arcs. A *digraph* (unlabelled) is an equivalence class of labelled digraphs under isomorphism.

Subgraphs

A *subgraph* of a digraph D is a digraph having all of its vertices and arcs in D. If D_1 is a subgraph of D, then D is a *supergraph* of D_1. A *spanning subgraph* of D is a subgraph containing all the vertices of D. For any set V_1 of vertices of D, the *induced subgraph* $\langle V_1 \rangle$ is the maximal subgraph of D with vertex set V_1. Thus $\langle V_1 \rangle$ contains all arcs of D incident with two vertices of V_1. The *removal* or *deletion* of vertex υ from D results in the induced subgraph $\langle V-\upsilon \rangle$ containing all vertices of V except υ and all arcs of D except those incident with υ. $\langle V-\upsilon \rangle$ is often written as $D-\upsilon$.

The degree of a vertex

$G(\upsilon) = \{u \mid uR\upsilon,\ u \in V\}$, the set of vertices of V adjacent to υ.

$id(\upsilon)$, the *indegree* of υ, is the number of arcs into υ.

$id(\upsilon) = \mid G(\upsilon) \mid$, the number of vertices dominating υ.

$L(\upsilon) = \{u \mid \upsilon Ru,\ u \in V\}$, the set of vertices of V adjacent from υ.

$od(\upsilon)$, the *outdegree* of υ, is the number of arcs from υ.

$od(\upsilon) = \mid L(\upsilon) \mid$, the number of vertices dominated by υ.

$N(\upsilon) = G(\upsilon) \cup L(\upsilon)$ is the *neighbourhood* of υ.

Types of digraph

A *graph* G is a symmetric digraph. The orientations of the arcs of a graph are normally omitted and the symmetric pairs of arcs are identified and called the *edges* of the graph.

An *oriented graph* is an asymmetric digraph, so $uR\upsilon$ precludes υRu.

A *tournament* T is a complete oriented graph, which is the same as an oriented complete graph. For distinct vertices u and υ, $u\upsilon \in T$ if and only if $\upsilon u \notin T$. For each vertex υ, $od(\upsilon) + id(\upsilon) = n - 1$.
Most of the above definitions follow [17] and [18].

Landau [27], [29] associated with each tournament T an ordered sequence of non-negative integers, its *score structure* L(T), formed by listing the vertex outdegrees in non-decreasing order.

Consider the following interpretation: n players take part in a round-robin tournament T, playing each other participant once in a match which cannot end in a tie and scoring a point for each win. Player υ is represented in T by vertex υ and "υ defeats u" by an arc υu. Then player υ gains $od(\upsilon)$ points and the vertex scores can be ordered to obtain the score structure of T.

Landau [27], [28], [29] interpreted tournaments as dominance relations in paired-comparison experi ments among animal societies. He studied score structures of tournaments and gave the condition for an arbitrary integer sequence to be the score structure of some tournament [29]. More recently [34], [17], [37], [18] tournament score structures have been called *score sequences*. We revert to the original phrase in order that the term "score sequence" can be reserved for an extension of the notion of a score structure.

In this chapter the concept of a score structure is extended to all oriented graphs. Let D be an oriented graph. Define $s_\upsilon = n - 1 + od(\upsilon) - id(\upsilon)$, the *score* of υ, so $0 \leq s_\upsilon \leq 2n - 2$. Let $V = \{1,2,\ldots,n\}$. Then, if $s_1 \leq s_2 \leq \ldots \leq s_n$, $S(D) = (s_1, s_2, \ldots, s_n)$ is the *score sequence* of D. In general, the *score sequence* of D is formed by listing the vertex scores in non-decreasing order.

The following modified interpretation of a tournament can be used: n players take part in a round-robin tournament D, playing each other participant once, with ties possible, scoring two points for each win and one point for each tie. "u ties with υ" would be represented in D by $uI\upsilon$. Player υ gains s_υ points and the vertex scores can be ordered to form the score sequence of D. Any oriented graph may be interpreted as such a "tournament". A simple example is the chess tournament [22, p. 49], where ties (draws) are common.

The following result showing that S(D) is a generalisation of L(T) is deduced directly from the definitions and corresponds to the fact that in a tournament score sequence two points are awarded for each win and ties are not possible.

<u>(1.1.1) Proposition</u>. For a tournament, $S(T) = 2L(T)$, that is $s_k = 2od(k)$ for all $k = 1, 2, \ldots, n$. //.

In section 1.2 we give the condition for an integer sequence to be the score sequence of some oriented graph. The proof provides an algorithm for the construction of an oriented graph $\tilde{D}(S)$ with given score sequence S. By constructing a tournament $\tilde{T}(L)$ with given score structure L, Landau's criterion for score structures of tournaments is deduced as a simple consequence. Section 1.3 is concerned with counting score sequences and labelled score sequences, where the scores have not been ordered. In section 1.4, as an introduction to the study of order relations in the following chapters, it is shown that $\tilde{D}(S)$ is a partial order so that every score sequence is the score sequence of some partial order. In section 1.5 we prove that there is no oriented graph with score sequence S which has fewer arcs than $\tilde{D}(S)$, and there is no tournament with score structure L which has more upsets than $\tilde{T}(L)$.

Finally, in section 1.6 it is shown that given two oriented graphs with the same score sequence, one can be transformed to the other by successively transforming appropriate intransitive triples to transitive triples or vice versa. This result is analogous to an earlier result of Ryser for tournaments.

SECTION 1.2 The criterion for a score sequence

(1.2.1) Theorem. Let

(1.2.2) $s_1 \leq s_2 \leq \dots \leq s_n$.

Then the integer sequence $S = (s_1, s_2, \dots, s_n)$ is the score sequence of some oriented graph if and only if

(1.2.3) $\sum_{i=1}^{n} s_i = n(n-1)$ and

(1.2.4) $\sum_{i=1}^{k} s_i \geq k(k-1)$ for $k = 1, 2, \dots, n-1$.

Proof. First note that (1.2.2), (1.2.3) and (1.2.4) together imply that $0 \leq s_k \leq 2n - 2$ for all $k = 1, 2, \dots, n$.

To show the necessity of conditions (1.2.3) and (1.2.4) we interpret the oriented graph D as a round-robin "tournament", as in the previous section. The n participants play a total of $\binom{n}{2}$ matches each of which contributes two points to $\sum_{i=1}^{k} s_i$. Thus (1.2.3) $\sum_{i=1}^{n} s_i = n(n-1)$.

Any k participants play $\binom{k}{2}$ matches in the "subtournament" induced by those k players and each match counts two points to the sum of their scores. So for any $V_1 \subseteq V$ with $|V_1| = k$,

$$\sum_{i \in V_1} s_i \geqslant k(k-1) \tag{1.2.5}$$

In particular (1.2.5) is true for those k players with least scores, and so

$$\sum_{i=1}^{k} s_i \geqslant k(k-1)$$

for $k = 1, 2, \ldots, n-1$ which is condition (1.2.4).

The sufficiency of (1.2.3) and (1.2.4) is proved by induction on n, being trivially true for $n = 1$. Suppose the theorem is true for sequences of order less than n and that (1.2.3) and (1.2.4) hold. It follows that $n - 1 \leqslant s_n \leqslant 2n-2$.

Define $x = 2n-2-s_n$, so $0 \leqslant x \leqslant n-1$, and let $n - x - p$ and $n - x + q$ be respectively the least and greatest indices for which

$$s_{n-x-p} = s_{n-x} = s_{n-x+q}, \quad 0 \leqslant p \leqslant n - x - 1, \quad 0 \leqslant q \leqslant x.$$

Now consider the sequence $S' = (s'_1, s'_2, \ldots, s'_{n-1})$ where

$$\begin{aligned} s'_k &= s_k \text{ for } k = 1, \ldots, n - x - p - 1 \\ &\qquad \text{and } k = n - x - p + q + 1, \ldots, n - x + q, \\ s'_k &= s_k - 1 \text{ for } k = n - x - p, \ldots, n - x - p + q \\ &\qquad \text{and } k = n - x + q + 1, \ldots, n - 1. \end{aligned} \tag{1.2.6}$$

S' is formed from S by deleting s_n and one point from each of x of the remaining vertex scores of S. As far as possible those vertices with greatest scores are chosen, but in the case of equal scores points are deleted from the vertices of least index so that the scores S' have the same ordering as in S.

Now $s'_1 \leqslant s'_2 \leqslant \ldots \leqslant s'_{n-1}$ so (1.2.2) is true for S'. We show that S' also satisfies (1.2.3) and (1.2.4) for $V' = \{1, 2, \ldots, n-1\}$, and is therefore by the induction hypothesis the score sequence of an oriented graph.

$s'_k = s_k - 1$ for x vertices of V'. Hence

$$\sum_{i=1}^{n-1} s'_i = \sum_{i=1}^{n-1} s_i - x = n(n-1) - s_n - x = (n-1)(n-2), \text{ and so}$$

(1.2.3) is true for S'.

To show that (1.2.4) is true for S', let

$$e_k = \sum_{i=1}^{k} s_i - k(k-1) \qquad \text{for } k = 1, 2, \ldots, n-1.$$

To prove (1.2.4) for S' it is sufficient to show that $e_k \geqslant x$ for $k = n - x, \ldots, n-1$ and $e_{n-x-p-1+r} \geqslant r$ for $r = 1, \ldots, p$.

By (1.2.4) for S it follows that $e_k \geqslant 0$ for all k and $e_{n-1} = x$.

$$\text{Now } x = e_{n-1} = \sum_{i=1}^{n-1} s_i - (n-1)(n-2)$$

$$= \sum_{i=1}^{k} s_i + \sum_{i=k+1}^{n-1} s_i - (n-1)(n-2)$$

$$\leqslant \sum_{i=1}^{k} s_i + (n-k-1)s_n - (n-1)(n-2)$$

$$= k(k-1) + e_k + (n-k-1)(2n-2-x) - (n-1)(n-2)$$

$$= e_k - (n-k-1)(k-n+x),$$

so $e_k \geqslant x$ for $k = n-x, \ldots, n-1$.

Also, for $r = 1, \ldots, p + q + 1$,

$$e_{n-x-p-1+r} = \sum_{i=1}^{n-x-p-1+r} s_i - (n-x-p-1+r)(n-x-p-2+r)$$

$$= \sum_{i=1}^{n-x-p-1} s_i + r\, s_{n-x} - (n-x-p-1+r)(n-x-p-2+r)$$

$$= (n-x-p-1)(n-x-p-2) + e_{n-x-p-1} + rs_{n-x} - (n-x-p-1+r)(n-x-p-2+r)$$

$$= e_{n-x-p-1} + rs_{n-x} - r(2n-2x-2p+r-3).$$

Letting $r = p+q+1$ and using $e_{n-x+q} \geqslant 0$ gives

$$s_{n-x} \geqslant 2n-2x-p+q-2 \quad - \quad \frac{e_{n-x-p-1}}{p+q+1} \quad ,$$

and so for $r = 1, \ldots, p+q+1$,

$$e_{n-x-p-1+r} \geqslant e_{n-x-p-1} \left(1 - \frac{r}{p+q+1}\right) + r + r(p+q-r)$$

which is $\geqslant r$ for $r = 1, \ldots, p$.

Thus $\sum_{i=1}^{k} s_i' \geqslant k(k-1)$ for $k = 1, 2, \ldots, n-2$ and so (1.2.4) is true for S'.

We can now give an algorithm for the recursive construction of an oriented graph $\tilde{D}(S)$ with score sequence S.

<u>(1.2.7) Algorithm</u>. Let $S=(s_1, s_2, \ldots, s_n)$ satisfy (1.2.2), (1.2.3) and (1.2.4). Construct an oriented graph $\tilde{D}(S)$ with score sequence S as follows:

Form S' as in (1.2.6). For $k = 1, 2, \ldots, n-1$,

nRk in $\tilde{D}(S)$ if $s_k' = s_k$

nIk in $\tilde{D}(S)$ if $s_k' = s_k-1$.

Then replace S by S', replace n by n-1 and proceed as before until $\tilde{D}(S)$ is completely defined.

To return to the proof of Theorem (1.2.1), we have shown that (1.2.2), (1.2.3) and (1.2.4) are true for S'. By the induction hypothesis, there is an oriented graph with score sequence S'. Now introduce vertex n into this oriented graph under the same conditions as in Algorithm (1.2.7) to form an oriented graph with score sequence S. //.

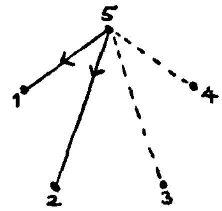
<u>(1.2.8) Example of Algorithm (1.2.7)</u>. $n = 5$, $S = (2, 3, 4, 5, 6)$.

$u \longrightarrow \upsilon$ represents $uR\upsilon$.

$u \cdots\cdots \upsilon$ represents $uI\upsilon$.

<u>Step 1</u> $S = (2, 3, 4, 5, 6)$

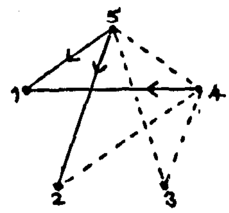
$\tilde{D}_1(S)$ is



$S' = (2, 3, 3, 4)$

<u>Step 2</u> $S = (2, 3, 3, 4)$

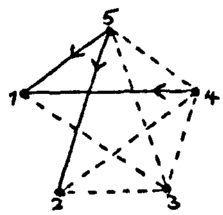
$\tilde{D}_2(S)$ is



$S' = (2, 2, 2)$

<u>Step 3</u> $S = (2, 2, 2)$

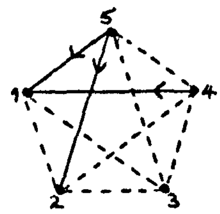
$\tilde{D}_3(S)$ is



$S' = (1, 1)$

<u>Step 4</u> $S = (1, 1)$

$\tilde{D}(S)$ is



$S' = (0)$

After the (n-1)th step, when $S' = (0)$, $\tilde{D}(S)$ is completely defined.

The following bound for s_k is analogous to a result of Landau [29, p.148] for score structures of tournaments.

<u>(1.2.9) Corollary</u>. If an oriented graph has score sequence $S = (s_1, s_2, \ldots, s_n)$ then

(1.2.10) $\quad k - 1 \leq s_k \leq n + k - 2 \qquad$ for $k = 1, 2, \ldots, n$

<u>Proof</u>. If $s_k < k - 1$ then

$$\sum_{i=1}^{k} s_i < k(k-1), \text{ contradicting condition (1.2.4).}$$

If $s_k > n + k - 2$ then $\sum_{i=k}^{n} s_i > (n-k+1)(n+k-2)$, so

$$\sum_{i=1}^{k-1} s_i < n(n-1) - (n-k+1)(n+k-2) = (k-1)(k-2), \text{ again contradicting (1.2.4).}$$

Equality is possible in either bound for

$$S = (s_1, s_2, \ldots, s_n) \text{ where } \begin{array}{l} s_1 = \ldots = s_k = k-1 \\ s_{k+1} = \ldots = s_n = n+k-1, \end{array}$$

and $$S = (s_1, s_2, \ldots, s_n) \text{ where } \begin{array}{l} s_1 = \ldots = s_{k-1} = k-2 \\ s_k = \ldots = s_n = n+k-2 \end{array}$$

both satisfy (1.2.2), (1.2.3) and (1.2.4). //.

(1.2.11) Remark. Although (1.2.10) is a necessary condition for a score sequence, it is not together with (1.2.2) and (1.2.3) sufficient. For example $S = (0, 1, 5, 6)$ satisfies $k-1 \leq s_k \leq k+2$ for $k = 1, 2, 3$ and 4 but violates (1.2.4) for $k = 2$.

We next consider strictly-increasing score sequences before giving, as a consequence of Theorem (1.2.1), a proof of the criterion for a score structure of a tournament. A score sequence S is *strictly-increasing* if $s_1 < s_2 < \ldots < s_n$.

(1.2.12) Lemma. Define

(1.2.13) $$l_k = s_k - (k-1) \qquad \text{for } k = 1, 2, \ldots, n.$$

Then $S = (s_1, s_2, \ldots, s_n)$ is a strictly-increasing score sequence if and only if

(1.2.14) $$l_1 \leq l_2 \leq \ldots \leq l_n$$

(1.2.15) $$\sum_{i=1}^{n} l_i = \tfrac{1}{2}n(n-1)$$

and (1.2.16) $$\sum_{i=1}^{k} l_i \geq \tfrac{1}{2}k(k-1) \qquad \text{for } k = 1, 2, \ldots, n-1.$$

Proof. Suppose S is strictly-increasing. Then

(1) $$l_k = s_k - (k-1) \leq s_{k+1} - k = l_{k+1} \text{ for } k = 1, 2, \ldots, n-1,$$

and (1.2.14) is true;

(2) $\sum_{i=1}^{n} l_i = \sum_{i=1}^{n} \{ s_i - (i-1) \} = \sum_{i=1}^{n} s_i - \frac{1}{2}n(n-1) = \frac{1}{2}n(n-1)$ by

(1.2.3), and (1.2.15) is true;

(3) $\sum_{i=1}^{k} l_i = \sum_{i=1}^{k} \{ s_i - (i-1) \} = \sum_{i=1}^{k} s_i - \frac{1}{2}k(k-1) \geq \frac{1}{2}k(k-1)$ for

$k = 1, 2, \ldots, n-1$ by (1.2.4), and (1.2.16) is true.

Conversely, let $L = (l_1, l_2, \ldots, l_n)$ satisfy (1.2.14), (1.2.15) and (1.2.16). Then

(1) $s_k = l_k + (k-1) < l_{k+1} + k = s_{k+1}$ for $k = 1, 2, \ldots, n-1$, and S is a strictly-increasing sequence;

(2) $\sum_{i=1}^{n} s_i = \sum_{i=1}^{n} \{ l_i + (i-1) \} = \sum_{i=1}^{n} l_i + \frac{1}{2}n(n-1) = n(n-1)$ by

(1.2.15), and (1.2.3) is true;

(3) $\sum_{i=1}^{k} s_i = \sum_{i=1}^{k} \{ l_i + (i-1) \} = \sum_{i=1}^{k} l_i + \frac{1}{2}k(k-1) \geq k(k-1)$ for

$k = 1, 2, \ldots, n-1$ by (1.2.16), and (1.2.4) is true.

So S is a strictly-increasing score sequence. //.

<u>(1.2.17) Theorem</u> (Landau [29], 1953). An integer sequence $L = (l_1, l_2, \ldots, l_n)$ for which condition (1.2.14) is true is the score structure of some tournament, where $l_k = od(k)$, if and only if it satisfies (1.2.15) and (1.2.16).

<u>Proof</u>. The necessity of (1.2.15) and (1.2.16) is shown in a similar manner to the necessity of (1.2.3) and (1.2.4) in Theorem (1.2.1).

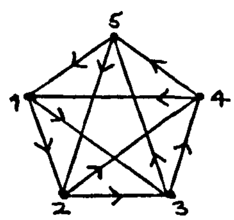
We prove that for any L satisfying (1.2.14), (1.2.15) and (1.2.16) there is a tournament $\tilde{T}(L)$ with score structure L. Let S be the strictly-increasing score sequence defined by (1.2.13). $\tilde{D}(S)$ has the property that no vertex k is adjacent from any of {1, 2, ..., k-1} for k = 1, 2, ..., n.

In $\tilde{D}(S)$ we shall now represent uRυ by two arcs uPυ, and uIυ by an arc uPυ and an arc υPu, so that each arc uPυ counts one point to u in S. Then there is an arc uPυ in $\tilde{D}(S)$ if u > υ.

For every pair of vertices u,υ in V with u > υ delete one arc uPυ from $\tilde{D}(S)$. The deleted arcs form a *transitive tournament.* This removes k - 1 points from the score of k and the remaining arcs form a tournament $\tilde{T}(L)$. Now in $\tilde{T}(L)$, od(k) = $s_k - (k-1) = l_k$ and $\tilde{T}(L)$ has score structure L. //.

In fact, to form $\tilde{T}(L)$ from $\tilde{D}(S)$, merely change uIυ, where u > υ, in $\tilde{D}(S)$ to υRu in $\tilde{T}(L)$. As an example consider the score structure L = (2, 2, 2, 2, 2). Then S = (2, 3, 4, 5, 6) as in Example (1.2.8) and, from $\tilde{D}(S)$, $\tilde{T}(L)$ is formed as below.

$\tilde{T}(2, 2, 2, 2, 2)$ is



Ryser [39, Theorem 4.1] has also given a recursive constructive proof of Theorem (1.2.17).

<u>(1.2.18) Ryser's Algorithm</u>. Let L = (l_1, l_2, ..., l_n) be a score structure.

Define a tournament $\tilde{R}(L)$ as follows:

n is dominated by the (n-1) - l_n vertices with greatest scores subject to the requirement that in the case of equal scores n is dominated by the vertices of least index. All other vertices are dominated by n. In the same way as (1.2.7), the algorithm preserves the ordering of the scores at each step so that if

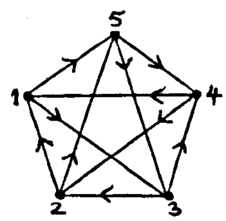
$L' = (l_1', l_2', \ldots, l_{n-1}')$ where $l_k' = l_k$ if nRk in $\tilde{R}(L)$

and $l_k' = l_k - 1$ if kRn in $\tilde{R}(L)$

then $l_1' \leq l_2' \leq \ldots \leq l_{n-1}'$. Next replace L by L', replace n by n-1 and continue until $\tilde{R}(L)$ is completely defined. $\tilde{R}(L)$ has score structure L.

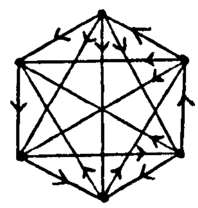
$\tilde{R}(L)$ is not in general the same as $\tilde{T}(L)$, as the case L = (2, 2, 2, 2, 2) proves.

$\tilde{R}(2, 2, 2, 2, 2)$ is

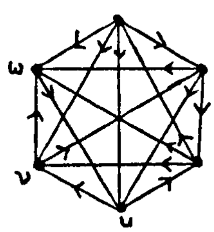


Further, $\tilde{T}(L)$ and $\tilde{R}(L)$ need not be isomorphic as unlabelled tournaments For example, with L = (1, 1, 2, 3, 3, 4)

$\tilde{T}(L)$ is



and $\tilde{R}(L)$ is



$\tilde{R}(L)$ is transformed to $\tilde{T}(L)$ if the orientations of the arcs of the *cyclic triple* uRυ, υRω, ωRu are reversed.

SECTION 1.3 Counting score sequences

It was shown in sections 1.1 and 1.2 that

(1) there is a one-to-one correspondence between the score structures and even score sequences of order n (Proposition (1.1.1))

(2) there is a one-to-one correspondence between the score structures and strictly-increasing score sequences of order n (Lemma (1.2.12) and Theorem (1.2.17)).

In this section we turn to counting the score sequences of order n, the number s(n) of integer sequences satisfying (1.2.2), (1.2.3) and (1.2.4). The recurrence method used is a refinement of an analogous method used by Narayana and Bent [34] to count score structures. In particular it reduces the computation and the range of parameters needed in the recurrence.

Let p(n, s, r) be the number of integer sequences $S = (s_1, s_2, \ldots, s_n)$ for which (1.2.2), (1.2.4),

(1.3.1) $$\sum_{i=1}^{n} s_i = s \qquad \text{where } s \geqslant n(n-1)$$

and (1.3.2) $$s_n = r \qquad \text{hold.}$$

<u>(1.3.3) Lemma</u>. $s(n) = \sum_{r=n-1}^{2n-2} p(n, n(n-1), r)$.

<u>Proof</u>. Condition (1.2.3) requires $s = n(n-1)$

and Corollary (1.2.9) requires $n - 1 \leqslant r \leqslant 2n - 2$. //.

The next two results establish a recurrence relation for p(n, s, r).

<u>(1.3.4) Lemma</u>. For $n \geqslant 2$,

$$p(n,\ n(n-1),\ r) = \sum_{r'=n-1}^{r} p(n-1,\ n(n-1) - r,\ r') \quad \text{for } r = n-1, \ldots, 2n-3,$$

and $p(n,\ n(n-1),\ 2n-2) = s(n-1)$.

Proof. The number of sequences counted by $p(n,\ n(n-1),\ r)$ with $s_{n-1} = r'$ is $p(n-1,\ n(n-1) - r,\ r')$. Since

$(s_1, s_2, \ldots, s_{n-2}) \leqslant s_{n-1} \leqslant s_n$, then $\frac{n(n-1) - r}{n-1} \leqslant s_{n-1} \leqslant r$.

If $n - 1 \leqslant r < 2(n-1)$, the range for r' is $n - 1 \leqslant r' \leqslant r$ as required.

If $r = 2n - 2$, then (1.2.10) for $k = n-2$ requires $r' \leqslant 2n - 4$ and so $n - 2 \leqslant r' \leqslant 2n - 4$. But then

$$p(n,\ n(n-1),\ 2n-2) = \sum_{r'=n-2}^{2n-4} p(n-1,\ n(n-1) - (2n-2),\ r')$$

$$= s(n-1) \text{ by Lemma (1.3.3).} \quad //.$$

(1.3.5) Lemma. For $n \geqslant 2$ and $s > n(n-1)$,

$$p(n,\ s,\ r) = p(n,\ s-1,\ r-1) + p(n-1,\ s-r,\ r).$$

Proof. Two cases of sequences $(s_1, s_2, \ldots, s_n)$ are distinguished.

Case 1. $s_{n-1} < r$.

Then $s_{n-1} \leqslant r - 1$. So $S' = (s_1, s_2, \ldots, s_{n-1}, r-1)$ satisfies (1.2.2), and also (1.2.4) since $s \rightarrow s - 1 \geqslant n(n-1)$.

As $r \rightarrow r - 1$ the number of sequences counted by $p(n,\ s,\ r)$ with $s_{n-1} < r$ is $p(n,\ s-1,\ r-1)$.

Case 2. $s_{n-1} = r$.

Then $S' = (s_1, s_2, \ldots, s_{n-1})$ satisfies (1.2.2) and (1.2.4). Since $n \rightarrow n - 1$, $s \rightarrow s - r$ and $r \rightarrow r$, the number of sequences counted by $p(n,\ s,\ r)$ with $s_{n-1} = r$ is $p(n-1,\ s-r,\ r)$. //.

This result is similar to an exercise in [33, p. 70] on score structures of tournaments.

(1.3.6) Lemma. In order to evaluate $s(n)$ for $n \leqslant m$, the required ranges for computation of $p(n,\ s,\ r)$ are

(1.3.9) $\quad n = 1, \ldots, m$

(1.3.10) $\quad s = n(n-1), \ldots, n(m-1)$ $\quad$ given n

and (1.3.11) $\quad r = \left[\frac{s-1}{n}\right] + 1, \ldots, \min\left\{ s - (n-1)(n-2), \left[\frac{m(m-1) - s}{m-n}\right] \right\}$

given $n < m$ and s

and $\quad r = m-1, \ldots, 2m-2$ when $n = m$ and $s = m(m-1)$.

The square brackets indicate the integer part of a number.

Proof.

The range of n follows immediately from Lemma (1.3.3). //.

The range of s given n. By (1.2.4), $s \geq n(n-1)$.

In the case $s_1 = \ldots = s_n = s_{n+1} = \ldots = s_m = m - 1$, then $s = n(m-1)$.

Further, as $n \leq m$ then, by (1.3.3), $p(n, s, r)$ is not required for

$s > m(m-1)$.

Now $(s_1, \ldots, s_n) \leq (s_{n+1}, \ldots, s_m)$ means $s_{n+1} + \ldots + s_m \geq \frac{(m-n)s}{n}$, and

so $m(m-1) \geq s_1 + \ldots + s_m \geq s + \frac{(m-n)s}{n}$ which yields $s \leq n(m-1)$. //.

The range of r given n and s.

$$r \geq \left[\frac{s-1}{n}\right] + 1 \quad \text{since } s_n \geq (s_1, \ldots, s_{n-1}).$$

If $n < m$, then $r \leq \left[\frac{m(m-1)-s}{m-n}\right]$ since $s_n \leq (s_{n+1}, \ldots, s_m)$ and

$r \leq s - (n-1)(n-2)$ from (1.2.4) for $k = n - 1$.

If $n = m$, then $s = m(m-1)$ by (1.3.10), and, by Corollary (1.2.9),

$m - 1 \leq r \leq 2m - 2$. //.

To evaluate $s(n)$ for $n = 1, \ldots, m$, we calculate $p(n, s, r)$ for the ranges given by (1.3.9), (1.3.10) and (1.3.11), using the recurrences (1.3.4) and (1.3.5) and the initial condition $p(1, s, s) = 1$ for $s = 0, \ldots, m - 1$. Then (1.3.3) is used to determine $s(n)$. The results of a computer enumeration of $s(n)$ and $p(n, n(n-1), r)$ for $n \leq 22$ and $n - 1 \leq r \leq 2n - 2$ are displayed in Table 1 of Appendix 1.

The complicated nature of the three-variable recurrence required and the large range of feasible values of the parameters n, s and r make hand computation of s(n) impracticable. The constraints imposed by condition (1.2.4) seem to render it unlikely that there is an appreciably simpler way of determining s(n) than that outlined above. It is (1.2.4) that effectively prevents evaluation by use of generating functions. The elementary enumeration by Riordan [37] of score structures involves generating functions but is erroneous. The results are in conflict with [34] and, for example, he counts the sequences (1, 1, 1, 1, 6, 6, 6, 6) and (1, 1, 1, 2, 5, 6, 6, 6) as score structures whereas they violate (1.2.16) for $k = 4$. Our method has been modified to count score structures and computer results obtained for $n \leq 30$ concur with [34]. They are displayed in Table 2 of Appendix 1.

Condition (1.2.2) allows S to be represented conveniently by writing the members of the sequence in non-decreasing order. If the scores are not ordered then we are considering labelled integer sequences satisfying (1.2.3) and (1.2.5), which is equivalent to permitting any labelling of a score sequence rather than the specific labelling of (1.2.2). For example, $L_1(S) = (s_1, s_2, s_3, s_4) = (1, 2, 5, 4)$ and $L_2(S) = (2, 4, 5, 1)$ are different labellings of the score sequence $S = (1, 2, 4, 5)$. The scores of the unlabelled sequence S have been permuted in different ways to form the labelled sequences $L_1(S)$ and $L_2(S)$. Each score sequence represents an isomorphism class of labelled score sequences and s(n) counts the isomorphism classes.

Let ls(n), the number of *labelled score sequences* of order n, be the number of integer sequences $(s_1, s_2, \ldots, s_n)$ for which (1.2.3) and (1.2.5) are true. A method of determining ls(n) will now be outlined.

Define $lp(n, s, r)$ to be the number of sequences satisfying (1.2.3), (1.2.5), (1.3.1) and

(1.3.12) $$\max_{\upsilon \in V} \{s_\upsilon\} = r .$$

Then similarly to Lemma (1.3.3) it can be shown that :

<u>(1.3.13) Lemma</u>. $ls(n) = \sum_{r=n-1}^{2n-2} lp(n, n(n-1), r).$ //.

The following recurrence relation for $lp(n, s, r)$ is true:

<u>(1.3.14) Lemma</u>. For $s < nr$,

$$lp(n, s, r) = \sum_k \sum_{r'} \binom{n}{k} lp(n-k, s-kr, r')$$

where the ranges of summation of k and r' are respectively

(1) those integer values of k for which $\max \{1, s-n(r-1)\} \leq k \leq n-1$ and $s - kr \geq (n-k)(n-k-1)$,

(2) $r' = \left[\frac{s-kr-1}{n-k}\right] + 1, \ldots, \min \{r-1, s-kr-(n-k-1)(n-k-2)\}.$

For $s = nr$, $lp(n, s, r) = 1$.

<u>Proof</u>. <u>Case 1</u>. $s < nr$.

Suppose that there are k vertices of V with score r, where $1 \leq k < n$ since $s < nr$. These k vertices can be labelled in $\binom{n}{k}$ ways. The number of labelled sequences for the other $n - k$ vertices is then $lp(n-k, s-kr, r')$ where r', $\frac{s-kr}{n-k} \leq r' < r$, is the greatest score amongst these $n - k$ vertices. The number of labelled sequences for all vertices is then $\binom{n}{k} lp(n-k, s-kr, r')$ and to calculate $lp(n, s, r)$ this expression is summed over all feasible values of k and r'.

The range of k. It has been established that $1 \leq k < n$. Also $\frac{s-kr}{n-k} \leq r' \leq r - 1$ implies $k \geq s - n(r-1)$.
Since (1.2.5) must hold for the $n - k$ vertices with scores less than or equal to r', then $s - kr \geq (n-k)(n-k-1)$ and we have the required range of k.

The range of r' given k. It has been shown that r' is an integer in the interval $\left[\frac{s-kr}{n-k}, r - 1\right]$. Since (1.2.5) holds for the $n - k - 1$

vertices with least scores, then $s - kr - r' \geq (n-k-1)(n-k-2)$ and the range of r' follows.

Case 2 . $s = nr$.

Then all vertices of V have score r and there is only one such labelled sequence. //.

To determine $ls(n)$ for $n = 1, \ldots, m$ we calculate $lp(n, s, r)$ for the same ranges as $p(n, s, r)$, given by (1.3.9), (1.3.10) and (1.3.11), by using the recurrence relation (1.3.14) and the initial condition $lp(1, s, s) = 1$ for $s = 0, \ldots, m - 1$. Then $ls(n)$ is computed from (1.3.13). Results for $n \leq 12$ and $n - 1 \leq r \leq 2n - 2$ appear in Table 1 of Appendix 1 and analogous results for labelled score structures when $n \leq 14$ are given in Table 2 of Appendix 1.

SECTION 1.4 $\tilde{D}(S)$ is a partial order

An oriented graph D may be considered as an order relation of one or more of the following types if it satisfies the appropriate axioms.

D is a *linear (total, complete) order, transitive tournament* if for all u, υ and ω in V

RI. $\upsilon I \upsilon$ (R is irreflexive)

R1. $uR\upsilon$ and $\upsilon R\omega$ imply $uR\omega$

and RC. $u \neq \upsilon$ implies $uR\upsilon$ or υRu (R is complete).

RI and R1 together imply

RA. $uR\upsilon$ precludes υRu (R is asymmetric)

and RT. For distinct u, υ and ω in V,

$uR\upsilon$ and $\upsilon R\omega$ imply $uR\omega$ (R is transitive).

D is a *weak order* [30, p.178] if RI, R1 and

IT. For distinct u, υ and ω in V,

$u I \upsilon$ and $\upsilon I \omega$ imply $u I \omega$ (I is transitive).

Hence in a weak order, I is an equivalence relation on V.

D is a *semiorder* [30], [42], [38] if RI,

S1. For all u, υ, ω and z in V,

$u R \upsilon$ and $\omega R z$ imply $u R z$ or $\omega R \upsilon$

and S2. For all u, υ, ω and z in V,

$u R \upsilon$ and $\upsilon R \omega$ imply $u R z$ or $z R \omega$.

Given RI, either of S1 or S2 implies RA and RT.

D is a *partial order* if RI and R1, so a partial order is an irreflexive, asymmetric and transitive relation.

These types of order relation are successively more general.

<u>(1.4.1) Theorem</u>. $\tilde{D}(S)$ is a partial order for any score sequence S.

<u>Proof</u>. Since $\tilde{D}(S)$, as defined in Algorithm (1.2.7), is an oriented graph, it is irreflexive and asymmetric. It remains to prove that $\tilde{D}(S)$ is transitive. The proof is by induction on n, the number of vertices. Suppose $\tilde{D}(S)$ is transitive for score sequences of order less than n, which is certainly true for n = 3. If $\tilde{D}(S)$ is intransitive for some score sequence S of order n then, since $\tilde{D}(S') = \tilde{D}(S)-n$ is transitive by the induction hypothesis, any intransitive triple includes vertex n. Let u, υ, n be such a triple, where $u < \upsilon < n$. Since $u < \upsilon < n$, then $u R \upsilon$, $u R n$ and $\upsilon R n$ are not possible in $\tilde{D}(S)$, and it must be that $n R \upsilon$, $\upsilon R u$ and $n I u$. But it follows from the definition of $\tilde{D}(S)$ that $n I u$ and $n R \upsilon$ for $u < \upsilon$ implies that $s_u = s_\upsilon$. Hence $s'_u = s'_\upsilon - 1$.

But $\tilde{D}(S)-n$ is transitive. Hence $\upsilon R u$ implies that $G(\upsilon) \subset G(u)$ and L(u)

$\subset L(\upsilon)$ in $\tilde{D}(S)-n$. Thus $s'_\upsilon \geqslant s'_u + 2$ which contradicts the conclusion that $s'_u = s'_\upsilon - 1$. Hence the intransitive triple $nR\upsilon$, υRu and nIu is not possible for any $u < \upsilon < n$, and $\tilde{D}(S)$ is transitive for all S by induction. //.

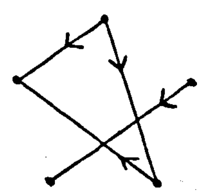
(1.4.2) Corollary. For any score sequence S, there is a partial order with score sequence S. //.

(1.4.3) Corollary. Let p(n) and pl(n) be respectively the numbers of partial orders and labelled partial orders (T_0 topologies) of order n. Then $s(n) \leqslant p(n)$ and $ls(n) \leqslant pl(n)$. //.

(It was shown in [6] that the number of labelled partial orders with n vertices is equal to the number of T_0 topologies with n points - see Proposition (4.1.2).)

Proof. $s(n) \leqslant p(n)$ by Corollary (1.4.2). Moreover, since the number of labellings of the score sequence $S \leqslant$ the number of labellings of the partial order $\tilde{D}(S)$, it follows that $ls(n) \leqslant pl(n)$. //.

$s(n) = p(n)$ for $n \leqslant 4$ only, as the following partial order shows. It has the same score sequence, S = (2, 3, 4, 5, 6), as the partial order $\tilde{D}(S)$ of Example (1.2.8).



(1.4.4)

$ls(n) = pl(n)$ for $n \leqslant 3$ only.

A computer enumeration of pl(n) for $n \leqslant 7$ appeared in [12] and a method of calculating pl(n) is described in [25]. The values of p(n) for $n \leqslant 6$ were stated in [4]. A computer search carried out by the author has verified these results and also evaluated p(7). These values are presented below together with the corresponding values of s(n) and ls(n).

(1.4.5) Table

n	s(n)	p(n)	ls(n)	pl (n)
1	1	1	1	1
2	2	2	3	3
3	5	5	19	19
4	16	16	201	219
5	59	63	3,081	4,231
6	247	318	62,683	130,023
7	1,111	2,045	1,598,955	6,129,859

The enumeration of p(n), see [10, p.493], [6, p.4], is an old and difficult graphical enumeration problem and no significant progress has yet been made on the problem.

Example (1.4.4) shows that the score sequence of a partial order need not determine that partial order up to isomorphism, so it is not a *complete set of invariants* of a partial order. The problem of specifying a complete set of invariants of a particular class of digraphs can be rephrased as:

(1.4.6) Given that D belongs to a certain class of digraphs, what information about D do we need in order to be able to completely determine D?

The *pair degree* of the vertex v is the ordered pair (od (v), id(v)).

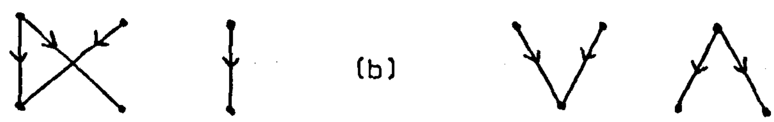
The sequence of ordered pairs (od(1), id(1)), ..., (od(n), id(n)) is the *degree sequence* of D. The *neighbourhood degree sequence* of vertex υ is the ordered pair of sequences

$$(od(x_1), id(x_1)), \ldots, (od(x_r), id(x_r)) \quad \text{where } L(\upsilon) = \{x_1, \ldots, x_r\}$$

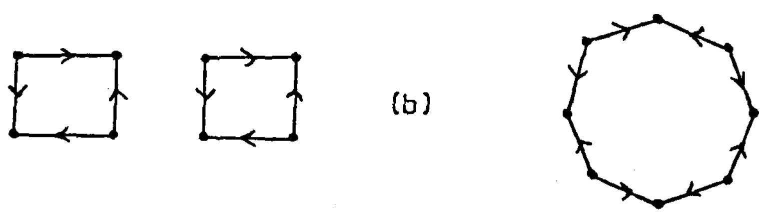
and $$(od(y_1), id(y_1)), \ldots, (od(y_s), id(y_s)) \quad \text{where } G(\upsilon) = \{y_1, \ldots, y_s\}.$$

The pairs of partial orders below show that in general neither the degree sequence nor the set of neighbourhood degree sequences of a partial order is sufficient to determine the partial order.

Same degree sequence {(2,0), (1,0), (1,0),(0,1), (0,1), (0,2)}

(a) (b)

(1.4.7)

Same set of neighbourhood degree sequences

(a) (b)

(1.4.8)

In (1.4.8) each partial order has

4 vertices with neighbourhood degree sequence {(0,2), (0,2)}, {ϕ}

and 4 vertices with neighbourhood degree sequence{ϕ}, {(2,0), (2,0)}.

Example(1.4.7) is a smallest such pair or partial orders, in that there is no pair with fewer vertices.

A problem related to (1.4.6) is:

(1.4.9) Given that D belongs to a certain class of digraphs and given a set of invariants, which digraphs in that class are completely determined by the given set of invariants?

This question is answered for score sequences of oriented graphs in section

2.2 and for score structures of tournaments in section 2.3.

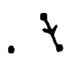
We return to problem (1.4.6) in section 3.2, where it is shown that a semiorder is completely determined by its score sequence, and in section 3.5 where reconstruction conjectures of Ulam and Harary, see [49, p.29], [16], [31], are proved for semiorders.

SECTION 1.5 Optimal properties of $\tilde{D}(S)$ and $\tilde{T}(L)$

(1.5.1) Lemma. Let D be an oriented graph with score sequence S which has a minimum number of arcs, so that no oriented graph with score sequence S has fewer arcs than D. Then D is a partial order.

Proof. Suppose that D is not a partial order. Then

either (a) D has an intransitive triple  which can be changed to , to form an oriented graph with the same score sequence and three fewer arcs than D,

or (b) D has an intransitive triple  which can be changed to , to form an oriented graph with the same score sequence and one arc fewer than D.

Hence the number of arcs in D is not a minimum unless D is transitive. //.

(1.4.2) is a corollary of Lemma (1.5.1), so this is an alternative way of proving that for every score sequence there is a partial order with that score sequence.

(1.5.2) Theorem. There is no oriented graph with score sequence S which has fewer arcs than $\tilde{D}(S)$.

Proof. The proof is by induction on n, the number of vertices. The theorem is clearly true for $n \leq 3$, and suppose it to be true for score sequences of order less than n. By Lemma (1.5.1), it is sufficient to prove that there is no partial order with score sequence S of order n which has fewer arcs than $\tilde{D}(S)$.

Let D be a partial order with score sequence S and let $D - n$ and $\tilde{D}(S) - n$ have score sequences $S^* = (s_1^*, s_2^*, \ldots, s_{n-1}^*)$ and $S' = (s_1', s_2', \ldots, s_{n-1}')$ respectively. Since D and $\tilde{D}(S)$ are partial orders there are no vertices adjacent to n, so $s_k^* = s_k - 1$ for $2n - 2 - s_n$ vertices of $\{1, 2, \ldots, n-1\}$ and $s_k^* = s_k$ for the remainder, and similarly for s_k'. S' has the property that if $s_u' = s_u - 1$ and $s_\upsilon' = s_\upsilon$, then $s_u \geq s_\upsilon$.

If $S^* = S'$, then, by assumption, $\tilde{D}(S') = \tilde{D}(S) - n$ contains no more arcs than $D - n$, and hence $\tilde{D}(S)$ contains no more arcs than D.

If S^* and S' are not identical, there are vertices u and υ with $s_u > s_\upsilon$, $s_u^* = s_u = s_u' + 1$ and $s_\upsilon' = s_\upsilon = s_\upsilon^* + 1$. By assumption, $\tilde{D}(S^*)$ contains no more arcs than $D - n$. We distinguish two cases for $\tilde{D}(S^*)$.

<u>Case (1)</u>. $uR\upsilon$ in $\tilde{D}(S^*)$.

If $uR\upsilon$ in $\tilde{D}(S^*)$, alter this arc to $uI\upsilon$ to form D_1 with score sequence S_1. Then $s_u^* \to s_u^* - 1 = s_u'$ and $s_\upsilon^* \to s_\upsilon^* + 1 = s_\upsilon'$, so S_1 differs from S' for two fewer indices than does S^*. D_1 has one arc fewer than $\tilde{D}(S^*)$. By assumption, the partial order $\tilde{D}(S_1)$ has no more arcs than D_1, so $\tilde{D}(S_1)$ has fewer arcs that $\tilde{D}(S^*)$ and $D - n$.

<u>Case (2)</u>. $uI\upsilon$ in $\tilde{D}(S^*)$.

Then since $s_u^* > s_\upsilon^*$ and $\tilde{D}(S^*)$ is transitive, there is a vertex ω in $\tilde{D}(S^*)$ with either $\omega R\upsilon$ and ωIu, or $uR\omega$ and $\upsilon I\omega$. If $\omega R\upsilon$ and ωIu, alter these arcs to ωRu and $\omega I\upsilon$ to form D_2. If $uR\omega$ and $\upsilon I\omega$, alter these arcs to $uI\omega$ and $\upsilon R\omega$ to form D_3. Now in forming D_2 or D_3 from $\tilde{D}(S^*)$, $s_u^* \to s_u^* - 1 = s_u'$, $s_\upsilon^* \to s_\upsilon^* + 1 = s_\upsilon'$ and the score of ω is unchanged.

So both D_2 and D_3 have the same score sequence S_1 as the oriented graph D_1 in case (1). D_2 and D_3 have the same number of arcs as $\tilde{D}(S^*)$. But, by assumption, $\tilde{D}(S_1)$ has no more arcs than D_2 or D_3. Therefore $\tilde{D}(S_1)$ has no more arcs than $\tilde{D}(S^*)$ or $D - n$.

Thus if $S^* \nsim S'$, there is a partial order $\tilde{D}(S_1)$ with no more arcs than $D - n$ whose score sequence differs from S' for two fewer indices than does S^*. If $S_1 \nsim S'$ we can replace S^* by S_1, pick appropriate vertices u and v, and proceed on $\tilde{D}(S_1)$ as before. In this way, we can form a sequence of partial orders, starting with $D - n$ and ending with $\tilde{D}(S')$, each of which has no more arcs than its predecessor. Hence $\tilde{D}(S') = \tilde{D}(S) - n$ contains no more arcs than $D - n$, so $\tilde{D}(S)$ contains no more arcs than D and the theorem is proved by induction. //.

$\tilde{D}(S)$ need not be a unique minimum. In Example (1.4.7), both partial orders have score sequence $S = (3, 4, 4, 6, 6, 7)$ and four arcs, and partial order (a) is $\tilde{D}(S)$. In this case, $S' = (3, 3, 4, 5, 5)$ and $S^* = (2, 4, 4, 5, 5)$.

Let T be a tournament with score structure L. Associate with T an oriented graph $D(T)$ in which uRv if $u > v$ and uRv in T, and uIv otherwise. Then $D(T)$ has score sequence S, where $s_k = l_k + (k-1)$. Also $D(\tilde{T}(L)) = \tilde{D}(S)$, where $\tilde{T}(L)$ is the tournament defined in the proof of Theorem (1.2.17).

An *upset* [45], [39, pp.115-116] in the tournament T is an arc uRv for which $u < v$. Ryser [39, Theorem 5.2] has provided a way of constructing a tournament with given score structure L which has a minimum number of upsets.

<u>(1.5.3) Lemma</u>. For any tournament T with n vertices, number of upsets in T + number of arcs in $D(T)$ = $\frac{1}{2}n(n-1)$.

Proof. Consider a pair of vertices u, υ where u < υ. There are $\frac{1}{2}n(n-1)$ such pairs. Either υRu or υIu in D(T). If there is an arc υRu in D(T), then υRu in T which is not an upset. If υIu in D(T), then uRυ in T which is an upset. //.

(1.5.4) Corollary. There is no tournament with score structure L which has more upsets than $\tilde{T}(L)$.

Proof. Let T be a tournament with score structure L. Let D(T) have score sequence S. Then $D(\tilde{T}(L)) = \tilde{D}(S)$. By Theorem (1.5.2), $\tilde{D}(S)$ has no more arcs than D(T). Hence by Lemma (1.5.3), T has no more upsets than $\tilde{T}(L)$. //.

Thus if the players are to be ranked according to the number of matches they have won, the proof of Theorem (1.2.17) provides a way of constructing a round-robin tournament $\tilde{T}(L)$ with a maximum number of upsets for players of prescribed abilities, defined by the score structure L.

SECTION 1.6 A transformation theorem

At the end of section 1.2, two tournaments $\tilde{T}(L)$ and $\tilde{R}(L)$ were constructed with score structure L = (1, 2, 2, 3, 3, 4), and it was remarked that $\tilde{R}(L)$ can be transformed to $\tilde{T}(L)$ by reversing the orientations of the arcs of a certain cyclic triple. This is an illustration of a result implicit in [39, Theorem 4.2].

(1.6.1) Theorem (Ryser, 1964). If two tournaments have the same score structure, then one can be transformed into the other by successively reversing the orientations of appropriate cyclic triples.

A *cycle* is a collection of distinct vertices $\upsilon_1, \upsilon_2, \ldots, \upsilon_n$ together with the arcs $\upsilon_1 R \upsilon_2, \upsilon_2 R \upsilon_3, \ldots, \upsilon_{n-1} R \upsilon_n, \upsilon_n R \upsilon_1$. The *length* of a cycle is the number of arcs (vertices) in the cycle. An *n-cycle* is a cycle of length n. A 3-cycle is called a *cyclic triple*. A digraph with no cycles is *acyclic*.

Actually, Ryser showed that a tournament can be transformed into any other tournament with the same score structure by successively reversing the orientations of the arcs of appropriate cyclic triples or 4-cycles. However, it is clear that any 4-cycle can be reversed by successively reversing two cyclic triples, so (1.6.1) is an immediate corollary.

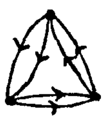
Kotzig [23] has generalised Theorem (1.6.1) to orientations with the same degree sequence of certain other graphs. (1.6.1) can also be extended, from score structures of tournaments to score sequences of oriented graphs, in the following way:

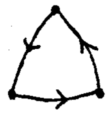
(1.6.2) Theorem. If D and D' are two oriented graphs with the same score sequence, then D can be transformed to D' by successively transforming appropriate triples in one of the following ways:

either (a) by changing an intransitive triple  to a transitive triple  with the same score sequence, or vice versa,

or (b) by changing an intransitive triple  to a transitive triple  with the same score sequence, or vice versa.

If we represent uRv, $u \rightarrow v$, by two arcs uPv, $u \rightleftarrows v$, and uIv, $u \cdot\cdot v$, by the two arcs uPv and vPu, $u \rightleftarrows v$, as in the proof of Theorem (1.2.17), then Theorem (1.6.2) may be rephrased as: D can be transformed to D' by successively transforming

either (a)  to  , that is by reversing the cyclic triple  , or vice versa,

or (b)  to  , that is by reversing the cyclic triple  , or vice versa.

So with this representation, Theorem (1.6.2) asserts that D can be transformed to D' by successively reversing the orientations of cyclic triples.

Before a proof of Theorem (1.6.2) can be given, it is necessary to introduce some preliminary definitions and results. It will be convenient to use the representation of $uR\upsilon$ and $uI\upsilon$ by two arcs, with $uP\upsilon$ indicating an arc from u to υ. Thus if $uR\upsilon$ there are two arcs $uP\upsilon$, and if $uI\upsilon$ then $uP\upsilon$ and υPu. With this representation, s_υ is the number of arcs from υ, the outdegree of υ, and the indegree of υ is $2n - 2 - s_\upsilon$.

(1.6.3) Lemma. If $s_u \geqslant s_\upsilon$ in D, then either $uP\upsilon$ or there is a vertex ω with $uP\omega$ and $\omega P\upsilon$.

Proof. Suppose that neither $uP\upsilon$ nor is there a vertex ω with $uP\omega$ and $\omega P\upsilon$. Then υRu and if $uR\omega$ or $uI\omega$, then $\upsilon R\omega$. Then $s_\upsilon \geqslant s_u + 2$, which contradicts $s_u \geqslant s_\upsilon$. Hence either $uP\upsilon$ or there is a vertex ω with $uP\omega$ and $\omega P\upsilon$. //.

(1.6.4) Corollary [33, p. 73]. If $l_u \geqslant l_\upsilon$ in the tournament T, then either $uR\upsilon$ or there is a vertex ω with $uR\omega$ and $\omega R\upsilon$.

Proof. If $l_u \geqslant l_\upsilon$, then $s_u = 2l_u \geqslant 2l_\upsilon = s_\upsilon$, and the result follows from (1.6.3) since $uP\upsilon$ in T implies that $uR\upsilon$ in T. //.

A *path* from υ_1 to υ_n in D is a collection of distinct vertices $\upsilon_1, \upsilon_2, \ldots, \upsilon_n$ together with the arcs $\upsilon_1 P\upsilon_2, \upsilon_2 P\upsilon_3, \ldots, \upsilon_{n-1} P\upsilon_n$. Notice that this definition differs from that in [17]. The *length* of a path is the number of arcs in the path. Lemma (1.6.3) says that if $s_u \geqslant s_\upsilon$ there is a path from u to υ of length at most two.

(1.6.5) Lemma. Let the oriented graph D have score sequence S and suppose that $u < v$, so $s_u \leq s_v$. Then there is a path from u to v if and only if

$$\sum_{i=1}^{k} s_i > k(k-1) \qquad \text{for } k = u, \ldots, v-1$$

(so that equality does not hold in condition (1.2.4) for any k, $u \leq k < v$).

Proof. If $\sum_{i=1}^{k} s_i > k(k-1)$ then there is an arc from $\{1, \ldots, k\}$ to $\{k+1, \ldots, n\}$ in D. Suppose $\omega P z$, where $\omega \leq k$ and $z \geq k+1$. Now since $s_k \geq s_\omega$ and $s_z \geq s_{k+1}$, then by Lemma (1.6.3) there are paths from k to ω and from z to k + 1. Hence there is a path from k to k+1. If $\sum_{i=1}^{k} s_i > k(k-1)$ for $k = u, \ldots, v-1$, then there is a path from k to k+1 for each $k = u, \ldots, v-1$, and so there is a subset of the arcs of these paths which forms a path from u to v in D.

Conversely, if $\sum_{i=1}^{k} s_i = k(k-1)$ for some k, where $u \leq k < v$, then there can be no arc from $\{1, \ldots, k\}$ to $\{k+1, \ldots, n\}$, and hence no path from u to v, in D. //.

(1.6.6) Corollary. Let D and D' have the same score sequence. Then there is a path from u to v in D if and only if there is a path from u to v in D'.

Proof. Suppose D and D' have score sequence S. Then by Lemmas (1.6.3) and (1.6.5) there is a path from u to v in D if an only if either $s_u \geq s_v$, or $s_u < s_v$ and

$$\sum_{i=1}^{k} s_i > k(k-1) \text{ for } k = u, \ldots, v-1.$$

The same is true for D'. //.

(1.6.7) Lemma . If there is a path from υ_1 to υ_n in D, then by successively reversing cyclic triples, it is possible to form an oriented graph with the same score sequence in which there is an arc $\upsilon_1 P \upsilon_n$.

Proof . Reversing a cyclic triple does not alter the score sequence of the resultant oriented graph. Let $\upsilon_1\ \upsilon_2 \ldots \upsilon_n$ be a shortest path from υ_1 to υ_n in D. Then there is no arc $\upsilon_i P \upsilon_n$ if $1 \leq i < n - 1$, for otherwise there would be a shorter path from υ_1 to υ_n. Hence for all $1 \leq i < n-1$ there is an arc $\upsilon_n P \upsilon_i$. In particular, there is an arc $\upsilon_n P \upsilon_{n-2}$. Reverse the cyclic triple $\upsilon_{n-2} P \upsilon_{n-1}$, $\upsilon_{n-1} P \upsilon_n$, $\upsilon_n P \upsilon_{n-2}$ to form D' with the same score sequence S as D in which $\upsilon_{n-2} P \upsilon_n$. Next reverse the cyclic triple $\upsilon_{n-3} P \upsilon_{n-2}$, $\upsilon_{n-2} P \upsilon_n$, $\upsilon_n P \upsilon_{n-3}$ in D' to form an oriented graph with score sequence S in which $\upsilon_{n-3} P \upsilon_n$. In this way, by successively reversing cyclic triples containing vertex n, we eventually form an oriented graph with score sequence S in which $\upsilon_1 P \upsilon_n$. //.

It follows from the proof of Lemma (1.6.7) that :

(1.6.8) Corollary . If there is a path from υ_1 to υ_n in D, then either $\upsilon_1 P \upsilon_n$ or D contains a cyclic triple $\upsilon_{n-2} P \upsilon_{n-1}$, $\upsilon_{n-1} P \upsilon_n$, $\upsilon_n\ P \upsilon_{n-2}$. //.

We can now give a proof of Theorem (1.6.2).

Proof of Theorem (1.6.2) . Suppose that the oriented graphs D and D' have the same score sequence S. Representing $uR\upsilon$ and $uI\upsilon$ by two arcs, it is sufficient to prove that D can be transformed to $\tilde{D}(S)$ by successively reversing the orientations of appropriate cyclic triples. For then D' can similarly be transformed to $\tilde{D}(S)$ and so $\tilde{D}(S)$ can be transformed to D' by the reverse sequence of transformations of triples. The proof is by induction on n, the number of vertices of D and D'. The theorem is clearly true for n = 1, 2 and 3. Suppose that it is true for oriented graphs with fewer than n vertices.

Now $\tilde{D}$ has the following properties:

(1) if $u > \upsilon$, then either $uR\upsilon$ or $uI\upsilon$, so there is an arc from u to υ, and

(2) if nIu and $nR\upsilon$, then $s_u \geqslant s_\upsilon$.

Suppose that the arcs between k and n have the same orientations in D and $\tilde{D}$ for $k = \upsilon + 1, \ldots, n - 1$ and that the orientations first differ for $k = \upsilon$. Since $nR\upsilon$ or $nI\upsilon$ in $\tilde{D}$, there are four possible cases:

(1) $nI\upsilon$ in D and $nR\upsilon$ in $\tilde{D}$

(2) υRn in D and $nR\upsilon$ in $\tilde{D}$

(3) $nR\upsilon$ in D and $nI\upsilon$ in $\tilde{D}$

and (4) υRn in D and $nI\upsilon$ in $\tilde{D}$.

In each case it will be shown that the orientations of the arcs between υ and n in D can be made to coincide with those in $\tilde{D}$ by reversing cyclic triples in D, without altering the orientations of the arcs between $\{\upsilon + 1, \ldots, n - 1\}$ and n.

<u>Case (1)</u>. $nI\upsilon$ in D and $nR\upsilon$ in $\tilde{D}$.

Since $id(n) = 2n - 2 - s_n$ is the same for D and $\tilde{D}$ there is some vertex $u < \upsilon$ which has more arcs to n in $\tilde{D}$ than in D. But by property (1) of $\tilde{D}$, as $n > u$ then nIu in $\tilde{D}$. Hence nRu in D. Now since nIu and $nR\upsilon$ in $\tilde{D}$ then by property (2) $s_u \geqslant s_\upsilon$, so in fact $s_u = s_\upsilon$.

As $s_u \geqslant s_\upsilon$ then by Lemma (1.6.3) there is a path of length at most two from u to υ in D. If $uP\upsilon$, reverse the cyclic triple $uP\upsilon$, υPn, nPu so that $nR\upsilon$ in the resultant oriented graph D_1. Otherwise υRu, so $uP\omega$ and $\omega P\upsilon$ for some ω. $\omega \neq n$ as nRu in D. Reverse the cyclic triple υPu, $uP\omega$, $\omega P\upsilon$ and then reverse the resultant cyclic triple $uP\upsilon$, υPn, nPu so that $nR\upsilon$ in the resultant oriented graph D_2.

Thus we have formed from D an oriented graph in which $nR\upsilon$ and nIu as in $\tilde{D}$.

Since the orientations of all other arcs of D incident with n are unaffected, then D_1 and D_2 agree with $\tilde{D}$ in the arcs between $\{\upsilon, \ldots, n-1\}$ and n. This proves case (1).

Case (2). υRn in D and $nR\upsilon$ in $\tilde{D}$.

Since id(n) is the same in D and $\tilde{D}$ there is some $u < \upsilon$ which has more arcs to n in $\tilde{D}$ than in D. Hence it follows from property (1)of $\tilde{D}$ that nIu in $\tilde{D}$ and nRu in D. Moreover, by property (2), $s_u \geq s_\upsilon$ as nIu and $nR\upsilon$ in $\tilde{D}$. Thus by Lemma (1.6.3), either $uP\upsilon$ in D, or υPu and there is a vertex ω with $uP\omega$ and $\omega P\upsilon$, where $\omega \neq n$ as υRn in D. If $uP\upsilon$, reverse the cyclic triple $uP\upsilon$, υPn, nPu to form D_1 in which υIn and nIu. Otherwise reverse the cyclic triple $uP\omega$, $\omega P\upsilon$ υPu and then reverse the resultant cyclic triple $uP\upsilon$, υPn, nPu to form D_2 in which υIn and nIu.

Now the orientations of all other arcs incident with n are the same in D_1 and D_2 as in D. Hence since υIn in D_1 and D_2, and $nR\upsilon$ in $\tilde{D}$, then case (2) can be reduced to case (1) which has already been proved.

Case (3). $nR\upsilon$ in D and $nI\upsilon$ in $\tilde{D}$.

Since id(n) is the same in D and $\tilde{D}$ there is some $u < \upsilon$ with uPn in D. As $u < \upsilon$, then $s_u \leq s_\upsilon$ and it follows from Lemma (1.6.3) that either υPu in D, or $uR\upsilon$ and there is a vertex ω with $\upsilon P\omega$ and ωPu. $\omega \neq n$ since $nR\upsilon$ in D. If υPu, reverse the cyclic triple υPu, uPn, $nP\upsilon$ to form D_1 in which $nI\upsilon$. If $uR\upsilon$, reverse the cyclic triple $\upsilon P\omega$, ωPu, $uP\upsilon$ and then reverse the resultant cyclic triple υPu, uPn, $nP\upsilon$ to form D_2 in which $nI\upsilon$.

Then $nI\upsilon$ in D_1 and D_2 as in $\tilde{D}$. Except that $uPn \to nPu$, the orientations of other arcs incident with n remain unaltered from D. So case (3) is proved.

<u>Case (4)</u>. υRn in D and $nI\upsilon$ in $\tilde{D}$.

Let s_k' be the score of vertex k in $D - n$ and $\tilde{s}_k'$ be the score of k in $\tilde{D} - n$, for $k = 1, \ldots, n - 1$. Then by assumption, $s_k' = \tilde{s}_k'$ for $k = \upsilon + 1, \ldots, n - 1$ and $s_\upsilon' = \tilde{s}_\upsilon' - 1$. Since $id(n)$ is the same in D and $\tilde{D}$ there is some $u < \upsilon$ with more arcs to n in $\tilde{D}$ than in D. Thus it follows from property (1) of $\tilde{D}$ that nIu in $\tilde{D}$ and nRu in D.

Let u be the vertex of greatest index less than υ for which nRu in D. Then for all $k = u + 1, \ldots, \upsilon$, either kRn or nIk in D, and either nIk or nRk in $\tilde{D}$. Hence there are as many arcs from k to n in D as in $\tilde{D}$. Thus $s_k' \leq \tilde{s}_k'$ for $k = u + 1, \ldots, \upsilon - 1$.

Hence $$\sum_{i=k}^{\upsilon} s_i' < \sum_{i=k}^{\upsilon} \tilde{s}_i' \qquad \text{for } k = u + 1, \ldots, \upsilon \tag{1.6.9}$$

Now, since $$\sum_{i=\upsilon+1}^{n-1} s_i' = \sum_{i=\upsilon+1}^{n-1} \tilde{s}_i'$$

and $$\sum_{i=1}^{n-1} s_i' = \sum_{i=1}^{n-1} \tilde{s}_i' = (n-1)(n-2),$$

then $$\sum_{i=1}^{\upsilon} s_i' = \sum_{i=1}^{\upsilon} \tilde{s}_i' .$$

Together with (1.6.9), this implies

$$\sum_{i=1}^{k-1} s_i' > \sum_{i=1}^{k-1} \tilde{s}_i' \geq (k-1)(k-2) \quad \text{for } k = u + 1, \ldots, \upsilon,$$

so that $$\sum_{i=1}^{k} s_i' > k(k-1) \text{ for } k = u, \ldots, \upsilon - 1.$$

Hence by Lemma (1.6.5) there is a path from u to υ in $D - n$.

Thus by Lemma (1.6.7) it is possible, by successively reversing cyclic triples in $D - n$, to transform D to D_1 with the same score sequence in which uPv. This does not alter the orientation in D of any arc incident with n. Now we reverse the cyclic triple uPv, vPn, nPu in D_1 to form D_2 in which nIv as in $\tilde{D}$. Hence case (4) is proved.

So by reversing cyclic triples in D, the orientations of the arcs between v and n in D can be made to coincide with those of $\tilde{D}$ without disturbing the orientations of the arcs between $\{v + 1, \ldots, n - 1\}$ and n. This can be done successively for vertices $v, v - 1, \ldots, 1$ to form D^* for which all arcs incident with n have the same orientations as in $\tilde{D}$. Then $D^* - n$ and $\tilde{D} - n$ have the same score sequence, and by assumption $D^* - n$ can be transformed to $\tilde{D} - n$ by reversing cyclic triples. So D can be transformed to $\tilde{D}$ and the theorem follows by induction. //.

The proof of Theorem (1.6.1) can be reduced to a particular case of the proof of Theorem (1.6.2).

Proof of Theorem (1.6.1). Let the tournament T have score structure L. It is sufficient to show that T can be transformed to $\tilde{T}(L)$ by successively reversing cyclic triples. Following the proof of Theorem (1.6.2), suppose that the arcs between $\{v + 1, \ldots, n - 1\}$ and n have the same orientations in T and $\tilde{T}$, but that the arcs between v and n are oriented differently. Then, either vRn in T and nRv in $\tilde{T}$, or nRv in T and vRn in $\tilde{T}$.

As in section 1.5, form an oriented graph $D(T)$ from T by defining uRv in $D(T)$ if $u > v$ and uRv in T, and uIv in $D(T)$ otherwise. Then $D(T)$ has score sequence S, where $s_1 < s_2 < \ldots < s_n$ since $s_k = l_k + (k-1)$, and $D(\tilde{T}(L)) = \tilde{D}(S)$. Now vRn in T and nRv in $\tilde{T}$ imply that nIv in $D(T)$ and

$nR\upsilon$ in $\tilde{D}$. As S is a strictly-increasing score sequence, then $nR\upsilon$ in $\tilde{D}$ implies that nRk in $\tilde{D}$ for all $k \leq \upsilon$. But since $nI\upsilon$ in D(T) and the arcs between $\{\upsilon + 1, \ldots, n - 1\}$ and n have the same orientations in D(T) and $\tilde{D}$, this implies that s_n in D(T) is less than s_n in $\tilde{D}$. Thus υRn in T and $nR\upsilon$ in $\tilde{T}$ leads to a contradiction and cannot arise. Thus $nR\upsilon$ in T and υRn in $\tilde{T}$, that is $nR\upsilon$ in D(T) and $nI\upsilon$ in $\tilde{D}$ which is case (3) in Theorem (1.6.2).

<u>Proof of the case $nR\upsilon$ in T and υRn in $\tilde{T}$.</u> Since $id(n) = n - 1 - l_n$ is the same in T and $\tilde{T}$ there is some $u < \upsilon$ with uRn in T and nRu in $\tilde{T}$. As $l_\upsilon \geq l_u$, it follows from Corollary (1.6.4) that either υRu in T, or $uR\upsilon$ and there is a vertex ω with $\upsilon R\omega$ and ωRu. $\omega \neq n$ since $nR\upsilon$ in T. If υRu then reverse the cyclic triple υRu, uRn, $nR\upsilon$ to form a tournament T_1 in which υRn. If $uR\upsilon$, reverse the cyclic triple $\upsilon R\omega$, ωRu, $uR\upsilon$, and then reverse the cyclic triple υRu, uRn, $nR\upsilon$ in the resultant tournament to form a tournament T_2 in which υRn. Then υRn and nRu in T_1 and T_2, as in $\tilde{T}$, and the orientations of the other arcs incident with n are unaltered. So the case $nR\upsilon$ in T and υRn in $\tilde{T}$ is proved. The proof is identical to the proof of case (3) in Theorem (1.6.2), with P replaced by R, D by T, s_k by l_k and Lemma (1.6.3) by Corollary (1.6.4).

So by reversing cyclic triples in T, it is possible to form a tournament T^* in which the orientations of the arcs incident with n coincide with those of $\tilde{T}$. The theorem then follows by induction, being clearly true for n = 1, 2, 3. //.

It is unlikely that the vertex scores have the same order in $T^* - n$ and $\tilde{T} - n$ as in T^* and $\tilde{T}$. This does not matter, for if necessary we can relabel the vertices of $T^* - n$ and $\tilde{T} - n$ before the next stage so that the scores are in non-decreasing order, then revert to the correct labelling after that stage has been completed, and so on.

CHAPTER 2. WEAK ORDERS, SIMPLE SCORE SEQUENCES AND SIMPLE SCORE STRUCTURES

SECTION 2.1 Introduction

A *paired comparison* experiment [21], [30], [9] is one in which n "objects" are presented in pairs to a "judge". For each pair of objects, the judge must state which he "prefers". More generally, the judge may be allowed to declare a tie, which may be interpreted as indifference between the two objects. If the objects are represented by vertices and the preference relation by R, then the *preference pattern* induced by the paired comparison experiment is an oriented graph if indifference is permitted and a tournament if indifference is not permitted. Paired comparison experiments are also used in other contexts: as remarked in section 1.1, Landau interpreted tournaments as dominance relations in paired comparison experiments among animal societies.

Kendall and Babington Smith [21] introduced the notion of *consistency (consistence)* in paired comparison experiments. If the judge has been consistent in his preferences, then it is reasonable to suppose that the preference relation R is transitive [30, p.178], in which case all triples are transitive. If indifference is not permitted, which is the case considered in [21], then all triples are of the form uRv, vRw, uRw and so a consistent preference pattern is a linear order. If indifference is permitted then one possible graphical interpretation of the notion of consistency [41, pp.17-21], [7, pp.104-105] is that indifference as well as preference should be transitive. In this case all triples are weakly ordered, and so a consistent preference pattern is a weak order. With this in mind we say that a score sequence S is *consistent* if every oriented graph with score sequence S is a weak order. S is *inconsistent* if no weak order has score sequence S. Otherwise, S is *partially consistent*.

A score sequence is *simple* if there is, up to isomorphism, exactly one oriented graph with that score sequence. Similarly, a score structure is *simple* [17, p.317] if there is, up to isomorphism, exactly one tournament with that score structure. In section 2.2 the criterion is given for a score sequence to be simple, so that for a given oriented graph we can decide whether it is specified up to isomorphism by its score sequence. We also determine the number of simple score sequences of order n and characterise consistent and partially consistent score sequences. In section 2.3 the criterion is derived for a tournament score structure to be simple, and the number of simple score structures of order n is evaluated.

SECTION 2.2 Weak orders and simple score sequences

(2.2.1) Theorem . Two further equivalent definitions of a weak order D are

(1) W1. $s_u > s_\upsilon$ implies $uR\upsilon$ in D

and W2. $s_u = s_\upsilon$ implies $uI\upsilon$ in D,

and (2) W3. $uR\upsilon$ in D implies $s_u > s_\upsilon$

and W4. $uI\upsilon$ in D implies $s_u = s_\upsilon$.

Proof .

(a) W1 and W2 imply D is a weak order

RI: By W2, $s_\upsilon = s_\upsilon$ implies $\upsilon I\upsilon$ in D. So RI is true.

R1: By W1 and W2, $uR\upsilon$ in D implies $s_u > s_\upsilon$ and $\upsilon R\omega$ in D implies $s_\upsilon > s_\omega$.

Hence $s_u > s_\omega$ and, by W1, $uR\omega$ in D. So R1 is true.

IT: By W1 and W2, $uI\upsilon$ in D implies $s_u = s_\upsilon$ and $\upsilon I\omega$ in D implies $s_\upsilon = s_\omega$. Hence $s_u = s_\omega$ and, by W2, $uI\omega$ in D. So IT is true.

Thus D is a weak order.

(b) <u>D is a weak order implies W3 and W4</u>

Suppose RI, R1 and IT are true for D.

W3: Let $uR\upsilon$. Then by R1, ωRu and $uR\upsilon$ imply $\omega R\upsilon$. But uIu and $uR\upsilon$. Hence $id(u) < id(\upsilon)$. Similarly, by R1, $\upsilon R\omega$ and $uR\upsilon$ imply $uR\omega$, and since $\upsilon I\upsilon$ and $uR\upsilon$, then $od(u) > od(\upsilon)$. Thus $s_u > s_\upsilon$ and W3 is true.

W4: Let $uI\upsilon$ and suppose ωRu. $\upsilon R\omega$ is not possible since, by R1, $\upsilon R\omega$ and ωRu imply υRu. $\upsilon I\omega$ is not possible since, by IT, $uI\upsilon$ and $\upsilon I\omega$ imply $uI\omega$. Thus ωRu implies $\omega R\upsilon$, and similarly $\omega R\upsilon$ implies ωRu. Hence $id(u) = id(\upsilon)$.

Now suppose that $uR\omega$. $\omega R\upsilon$ is not possible since, by R1, $uR\omega$ and $\omega R\upsilon$ imply $uR\upsilon$. $\upsilon I\omega$ is not possible since, by IT, $uI\upsilon$ and $\upsilon I\omega$ imply $uI\omega$. Thus $uR\omega$ implies $\upsilon R\omega$, and similarly $\upsilon R\omega$ implies $uR\omega$. Hence $od(u) = od(\upsilon)$. So altogether, $s_u = s_\upsilon$ and W4 is true.

(c) <u>W3 and W4 imply W1 and W2</u>

W1: Let $s_u > s_\upsilon$. Then, by W3, υRu implies $s_\upsilon > s_u$ which is not true and, by W4, $uI\upsilon$ implies $s_u = s_\upsilon$ which is also not true. Hence υRu and $uI\upsilon$ are not possible, so $uR\upsilon$ and W1 is true.

W2: Let $s_u = s_\upsilon$. Then, by W3, $uR\upsilon$ implies $s_u > s_\upsilon$ which is not true, and υRu implies $s_\upsilon > s_u$ which is not true. Hence $uR\upsilon$ and υRu are not possible, so $uI\upsilon$ and W2 is true. //.

<u>(2.2.2) Corollary</u>. The weak order D is identical to the weak order D' induced by its score sequence, defined by $uR\upsilon$ in D' if and only if $s_u > s_\upsilon$.

<u>Proof</u>. Let D be a weak order. By Theorem (2.2.1), W3 and W4 are true for D. By W3, $uR\upsilon$ in D implies $s_u > s_\upsilon$ which, by definition,

implies $uR\upsilon$ in D'. By W4, $uI\upsilon$ in D implies $s_u = s_\upsilon$ which, by definition, implies $uI\upsilon$ in D'. Hence $D = D'$. //.

(2.2.3) Corollary. There is at most one weak order with given score sequence S.

Proof. Let D and D' be weak orders with score sequence S.
By Theorem (2.2.1), W1 and W2 are true for D and D'.
By W1, $s_u > s_\upsilon$ implies $uR\upsilon$ in D and D'.
By W2, $s_u = s_\upsilon$ implies $uI\upsilon$ in D and D'.
Hence $D = D'$ and the result follows. //.

From Corollary (2.2.3) and the definitions of consistent and simple score sequences, it follows that :

(2.2.4) Corollary. A consistent score sequence is simple. //.

Next we derive the criterion for a score sequence to be simple. The vertex υ is *reachable* from vertex u if there is a path from u to υ in D. An oriented graph is *strongly connected* or *strong* if for every two vertices u and υ there is a path from u to υ and vice versa. A *strong component* of an oriented graph is a maximal strong subgraph; it is the subgraph induced by a maximal set of mutually reachable vertices.

In order to derive the criterion for a score sequence to be simple, it is necessary to consider the score sequences of strong components of oriented graphs. Let D be an oriented graph with score sequence S. S is *strong* if D is strong, see (1.6.6). The *strong components* of S are the score sequences of the strong components of D. The following result shows that the strong components of S are determined by the successive values of k for which $\sum_{i=1}^{k} s_i = k(k-1)$, that is the successive values of k for which equality holds in condition (1.2.4).

(2.2.5) Lemma . Let D have score sequence S. Suppose that

(1) $\sum_{i=1}^{p} s_i = p(p-1)$

(2) $\sum_{i=1}^{q} s_i = q(q-1)$

and (3) $\sum_{i=1}^{k} s_i > k(k-1)$ for $k = p + 1, \ldots, q - 1$,

where $0 \leq p < q \leq n$.

Then the subgraph induced by $\{p + 1, \ldots, q\}$ is a strong component of D with score sequence $(s_{p+1} - 2p, s_{p+2} - 2p, \ldots, s_q - 2p)$.

Proof . From (1) it follows that there is no path from a vertex of $\{1, \ldots, p\}$ to a vertex of $\{p + 1, \ldots, n\}$. From (2) it follows that there is no path from a vertex of $\{1, \ldots, q\}$ to a vertex of $\{q + 1, \ldots, n\}$. Hence any strong component of D containing a vertex of $\{p + 1, \ldots, q\}$ must be wholly contained within that set.

Now let $p + 1 \leq u < \upsilon \leq q$. Since $s_\upsilon \geq s_u$, then by Lemma (1.6.3) there is a path from υ to u in D. Also, by condition (3), it follows from Lemma (1.6.5) that there is a path from u to υ in D. Hence u and υ are in the same strong component of D and so the subgraph induced by $\{p + 1, \ldots, q\}$ is a strong component of D. Clearly, this strong component has score sequence $(s_{p+1} - 2p, s_{p+2} - 2p, \ldots, s_q - 2p)$. //.

As an illustration of Lemma (2.2.5), consider the score sequence $S = (2,2,2,8,8,10,10,14,18,18,20,22,22)$.

$$\sum_{i=1}^{k} s_i = k(k-1) \qquad \text{for } k = 3, 7, 8 \text{ and } 13.$$

Therefore the strong components of S are, in ascending order, (2,2,2) , (2,2,4,4) , (0) and (2,2,4,6,6).

(2.2.6) Lemma . The score sequence S is simple if and only if every strong component of S is simple.

Proof The two observations made at the beginning of the proof of Lemma (2.2.5) clearly generalise to: if u is in a higher-ranked strong component than υ in the oriented graph D, then $uR\upsilon$ in D. Thus an oriented graph is wholly determined by its strong components and their rankings relative to each other, and the result follows. //.

(2.2.7) Theorem . Let S be a strong score sequence. Then S is simple if and only if it is either (0) or (1,1).

Proof . By inspection, the strong score sequences of orders one and two, (0) and (1,1) respectively, are simple. Suppose S is strong and contains three vertices $u < \upsilon < \omega$. Let the strong oriented graph D have score sequence S. By Corollary (1.6.8), either $uI\upsilon$, $uI\omega$ and $\upsilon I\omega$ in D, in which case $uP\upsilon$, $\upsilon P\omega$ and ωPu, or D contains a cyclic triple, say $u'P\upsilon'$, $\upsilon'P\omega'$, and $\omega'Pu'$. Hence D contains a cyclic triple. Reversing the orientations of the arcs of this triple forms an oriented graph D_1 with score sequence S which has a different number of arcs from D, since reversing a cyclic triple either transforms a transitive triple to an intransitive triple or vice versa. So D_1 is not isomorphic to D and thus S is not simple. //.

The criterion for a score sequence to be simple is given by the following corollary of Lemma (2.2.6) and Theorem (2.2.7):

(2.2.8) Corollary . The score sequence S is simple if and only if every strong component of S is either (0) or (1,1); that is

$$\sum_{i=1}^{k} s_i = k(k-1) \quad \text{or} \quad \sum_{i=1}^{k+1} s_i = (k+1)k \quad \text{for } k = 1, 2, \ldots, n-2,$$

so that in condition (1.2.4) inequality does not hold for any two consecutive values of k. //.

Thus it is possible to decide whether a given score sequence is simple by using Lemma (2.2.5) to determine the strong components of S and then applying Corollary (2.2.8).

We conclude with three propositions characterising consistent and partially consistent score sequences and counting the simple score sequences of order n.

<u>(2.2.9) Proposition</u> . A score sequence is consistent if and only if it is simple.

<u>Proof</u> . By Corollary (2.2.4), a consistent score sequence is simple. Let the oriented graph D have simple score sequence S. A simple score sequence has the property that every strong component is either (0) or (1,1). Thus u and v are in the same strong component if and only if $s_u = s_v$. Hence if $s_u > s_v$, then u and v are in different strong components and so uRv in D. If $s_u = s_v$, u and v are in the same strong component which must then have score sequence (1,1), and so uIv in D. Thus by Theorem (2.2.1), D is a weak order, and S is consistent. //.

<u>(2.2.10) Proposition</u> . The number of simple (consistent) score sequences of order n is f(n), the nth number of the Fibonacci sequence, defined by $f(n) = f(n-1) + f(n-2)$ for $n \geq 3$, $f(1) = 1$ and $f(2) = 2$.

<u>Proof</u> . By Corollary (2.2.8), the score sequence S of order n is simple if and only if every strong component is either (0) or (1,1). The number of simple score sequences of order n with highest-ranked component (0) is f(n-1), and the number of simple score sequences of order n with highest-ranked component (1,1) is f(n-2). Since $f(1) = 1$ and $f(2) = 2$, by inspection, the result follows. //.

<u>(2.2.11) Proposition</u> . The score sequence S is partially consistent if and only if every strong component of order r has score sequence

$(r-1, r-1, \ldots, r-1)$, and there is at least one component of order greater than two.

Proof . If every strong component of the score sequence S is of order one or two then it is either (0) or (1,1) and hence, by Corollary (2.2.8), it is consistent. Thus if S is partially consistent it has a strong component of order greater than two.

Let S be a partially consistent score sequence and let D be the weak order with score sequence S. If $s_u > s_v$ then, by W1, uRv and so any two vertices with different scores are in different strong components of D. Thus every strong component in S of order r must have score sequence $(r-1, r-1, \ldots, r-1)$.

Conversely, suppose every strong component of order r has score sequence $(r-1, r-1, \ldots, r-1)$. This implies that if $s_u = s_v$ then u and v are in the same component of S, and if $s_u > s_v$ then u is in a higher-ranked component than v. Hence we can define an oriented graph D with score sequence S by uIv in D if $s_u = s_v$, and uRv in D if $s_u > s_v$. By Theorem (2.2.1), D is a weak order, and hence S is partially consistent. //.

SECTION 2.3 Simple score structures

In section 2.1 we defined a score structure as simple if it belongs to exactly one tournament. Every score structure of tournaments with fewer than five vertices is simple, [9, p.103], [17, p.317], but the following non-isomorphic tournaments both have score structure (1,1,2,3,3).

The objective of this section is to determine which tournaments are specified up to isomorphism by their score structures, that is which score structures are simple. In order to characterise simple score structures we consider the score structures of strong components of tournaments. The method is similar to that used in section 2.2 to determine simple score sequences, and it is necessary to restate some results of the previous section in the language of tournaments and score structures.

Let T be a tournament with score structure L. L is *strong* if T is strong. The *strong components* of L are the score structures of those tournaments which are the strong components of T. The following corollary to Lemma (2.2.5) shows that the strong components of L are determined by the successive values of k for which $\sum_{i=1}^{k} l_i = \frac{1}{2}k(k-1)$, that is the successive values of k for which equality holds in condition (1.2.16).

<u>(2.3.1) Corollary</u>. Let the tournament T have score structure L. Suppose that

(1) $$\sum_{i=1}^{p} l_i = \tfrac{1}{2}p(p-1)$$

(2) $$\sum_{i=1}^{q} l_i = \tfrac{1}{2}q(q-1)$$

and (3) $$\sum_{i=1}^{k} l_i > \tfrac{1}{2}k(k-1) \quad \text{for } k = p+1, \dots, q-1,$$

where $0 \leq p < q \leq n$.

Then the tournament induced by $\{p+1, \ldots, q\}$ is a strong component of T with score structure $(l_{p+1} - p, l_{p+2} - p, \ldots, l_q - p)$.

Proof . Putting $l_k = \frac{1}{2}s_k$ in Lemma (2.2.5) gives the required result. //.

Notice that a path from υ_1 to υ_n in T is a collection of distinct vertices $\upsilon_1, \upsilon_2, \ldots, \upsilon_n$ together with the arcs $\upsilon_1 R \upsilon_2, \upsilon_2 R \upsilon_3, \ldots, \upsilon_{n-1} R \upsilon_n$. The strong components of L can be derived using Corollary (2.3.1) in the same way as the strong components of a score sequence are derived using Lemma (2.2.5).

The next corollary is a special case of Lemma (2.2.6).

(2.3.2) Corollary . The score structure L is simple if and only if every strong component of L is simple. //.

It now remains to determine which strong score structures are simple.

(2.3.3) Theorem . Let L be a strong score structure. Then L is simple if and only if it is one of

$$(0) , (1,1,1) , (1,1,2,2) \text{ or } (2,2,2,2,2).$$

Proof . Let L be a strong score structure.

Three cases will be distinguished.

(1) L has at least three different scores;

(2) L has exactly two different scores;

(3) All scores of L are equal, in which case L is *regular*.

A tournament with a score structure of type (3) is called *regular* or *Eulerian*.

Case (1) . L has at least three different scores, say $l_u > l_\upsilon > l_\omega$. Then there is a tournament having score structure L with u, υ, ω as a cyclic triple. For suppose the tournament T has score structure L and that the triple u, υ, ω is not cyclic in T. Without loss of generality, let uRv, $vR\omega$ and $uR\omega$. As T is strong there is a path from ω to u (which may or may not include the vertex v) which together with the arc $uR\omega$ forms a cycle. Reverse the orientation of every arc of this cycle to form a tournament T_1 with score structure L, in which uRv, $vR\omega$ and ωRu so that u, v and ω form a cyclic triple.

Now reverse this cyclic triple to form the tournament T_2 with score structure L. T_2 is not isomorphic to T_1. For consider the number of arcs for which the score of the dominant vertex is less than the score of the dominated vertex. Since $l_u > l_v > l_\omega$, reversing the cyclic triple uRv, $vR\omega$, ωRu to form T_2 increases the number of such arcs by one. Therefore T_1 and T_2 are not isomorphic and so L is not simple.

Hence there is no simple strong score structure of type (1).

Case (2). L has exactly two different scores, say

$$l_1 = \dots = l_k < l_{k+1} = \dots = l_n \text{ for some } k.$$

Put $A = \{1, \dots, k\}$ and $B = \{k+1, \dots, n\}$.

Now $kl_1 + (n-k)l_n = \sum_{i=1}^{n} l_i = \frac{1}{2}n(n-1)$.

Since L is strong, $1 \leqslant l_1 < l_n$. Thus the above equation implies that $(n-k)l_n < \frac{1}{2}n(n-1) < nl_n$ and so

$$\tfrac{1}{2}(n-1) < l_n < \tfrac{1}{2}\frac{n(n-1)}{n-k} .$$

Hence $k \neq 1$ because l_n must be an integer and there is no integer between $\frac{1}{2}(n-1)$ and $\frac{1}{2}n$. Thus $k \geqslant 2$ and so $|A| \geqslant 2$. Similarly, $|B| \geqslant 2$.

Let T be a tournament with score structure L. Suppose that the tournament T(A) induced by A has score structure $L(T(A)) = (a_1, a_2, \dots, a_k)$ where $a_1 \leqslant a_2 \leqslant \dots \leqslant a_k$. Because $a_1 \leqslant a_2 \leqslant \dots \leqslant a_k$ and $l_1 = l_2 = \dots = l_k$, then vertex 1 dominates as many vertices of B as any of $\{2, \dots, k\}$, and vertex k is dominated by as many vertices of B as any of $\{1, \ldots, k-1\}$. Since T is strong, there are arcs from A to B and from B to A. Hence 1 dominates some vertex u of B and k is dominated by some vertex υ of B, where υ is not necessarily different from u.

As $a_k \geqslant a_1$, then by Corollary (1.6.4) there is a path $k, j, 1$ in T(A) from k to 1 of length at most two. Similarly, as $l_u = l_\upsilon$ there is a path u, ω, υ in T from u to υ of length at most two. It follows that the vertices $l, u, \omega, \upsilon, k, j$ form a cycle in T of length at most six.

The orientations of the arcs of this cycle can be reversed to form another tournament T_1 with score structure L. Let $L(T_1(A))$ denote the score structure of the subtournament of T_1 induced by A. Now in forming $T_1(A)$ from $T(A)$

$$a_1 \to a_1 + 1$$
$$a_k \to a_k - 1$$
$$a_j \text{ is unchanged if } j \neq 1 \text{ or } k$$
$$\text{and } a_\omega \text{ is unchanged if } \omega \in A.$$

Therefore $L(T_1(A)) \neq L(T(A))$ unless $a_k = a_1 + 1$. But if $L(T_1(A)) \neq L(T(A))$ then $T_1(A)$ is not isomorphic to $T(A)$, in which case T_1 is not isomorphic to T and L is not simple. So L can be simple only if $a_k = a_1 + 1$, that is if k is even, $k = 2r$ say, and

$$a_1 = \ldots = a_r = r - 1$$
$$a_{r+1} = \ldots = a_{2r}(a_k) = r\ .$$

Suppose $L(T(A))$ is of this form. Then for some value of q, each vertex of $\{1, \ldots, r\}$ dominates q vertices of B and each vertex of $\{r + 1, \ldots, 2r\}$ dominates $q - 1$ vertices of B so that $l_1 = \ldots = l_{2r} = r + q - 1$. Now $q \geqslant 1$ since L is strong. Moreover,

$$l_{2r+1} = \dots = l_n \text{ and } \sum_{i=1}^{n} l_i = \tfrac{1}{2}n(n-1), \text{ so}$$

$$\tfrac{1}{2}n(n-1) = 2r(r+q-1) + (n-2r)\, l_n \,.$$

This equation and the inequality $l_n > l_{2r} = r + q - 1$ together imply that $q < n - 2r - \frac{1}{2}(n-2r-1)$.

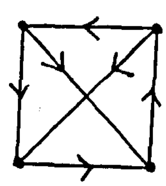
As $n - 2r = |B| \geq 2$, then $n - 2r - 1 > 0$ and so $q < n - 2r$.

Therefore each vertex of $\{1, \dots, r\}$ is dominated by at least one vertex of B.

Now suppose that $r \geq 2$, so $|A| \geq 4$, and let the vertex 2 be dominated by vertex υ of B. Then arguing the same way as in the general case, there is a cycle 1, u, ω, υ, 2, j in T of length at most six, where $j \in A$. This cycle can be reversed to form a tournament T_1 with score structure L. In this case $a_2 \to r - 2$.

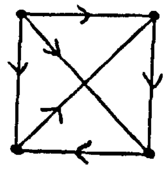
Therefore $L(T_1(A)) \neq L(T(A))$ since the least score in T(A) is $r - 1$ whereas there is a score of $r - 2$ in $T_1(A)$. Hence $T_1(A)$ is not isomorphic to T(A), so T_1 is not isomorphic to T and L is not simple. Thus L is not simple if $r \geq 2$.

Altogether, if $|A| \geq 3$ then L is not simple. Similarly, if $|B| \geq 3$ then L cannot be simple. Since $|A| \geq 2$ and $|B| \geq 2$, the only remaining possibility is $|A| = |B| = 2$. The only strong score structure of type (2) with $|A| = |B| = 2$ is $L = (1,1,2,2)$ which is simple. The tournament with this score structure is:



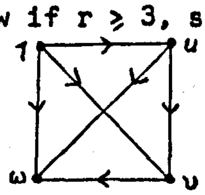
So in case (2) the only simple strong score structure is (1,1,2,2).

Case (3) . All scores of L are equal, so the number of vertices, n, is odd. Let $n = 2r + 1$. Define T to be the tournament with vertices $\{1, 2, \ldots, 2r+1\}$ in which for all $k \in \{1, 2, \ldots, 2r+1\}$, the vertex k dominates vertices $k + 1, k + 2, \ldots, k + r$ reduced modulo $2r + 1$. Then L(T) has all scores equal.

T has no subtournaments T_4^* of the type  .

For suppose without loss of generality that vertex 1 dominates vertices u, υ and ω in T, where $2 \leqslant u < \upsilon < \omega \leqslant r + 1$. Then, by the definition of T, the triple u, υ, ω is transitive with uRυ, υRω and uRω. So T cannot have any subtournatments T_4^*.

Now if $r \geqslant 3$, so $n \geqslant 7$, then T has a transitive subtournament

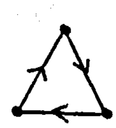


induced by the distinct vertices 1, u, υ and ω.

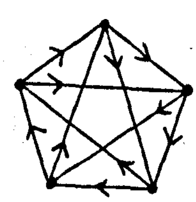
Let z be the vertex $\omega + r$. Then uRω, ωRz and zRu in T. Reversing the cyclic triple u, ω, z forms another tournament T_1 with score structure L. But T_1 has a subtournament of type T_4^* induced by the vertices 1, u, υ and ω, and is therefore not isomorphic to T. So L is not simple if $n \geqslant 7$.

The regular score structures of orders 1, 3 and 5 are simple, the associated Eulerian tournaments being respectively:

L = (0)



L = (1,1,1)



L = (2,2,2,2,2).

So the only simple strong score structures of type (3) are (0), (1,1,1) and (2,2,2,2,2).

(2.3.4) Corollary. The score structure L is simple if and only if every strong component of L is one of

$$(0)\ ,\ (1,1,1)\ ,\ (1,1,2,2) \text{ or } (2,2,2,2,2)\ . \qquad //.$$

Thus it is possible to decide whether a given score structure L is simple by using Corollary (2.3.1) to determine the strong components of L and then applying Corollary (2.3.4).

Let $g(n)$ denote the number of simple score structures of order n. $g(n)$ is easily evaluated using the following recurrence relation.

(2.3.5) Proposition. $g(n) = g(n-1) + g(n-3) + g(n-4) + g(n-5)$,

where $g(k) = 0$ if $k < 0$ and $g(0) = 1$.

Proof. By Corollary (2.3.2), a score structure is simple if and only if every strong component is simple. If the highest-ranked component is of order k, then letting $h(k)$ denote the number of strong simple score structures of order k,

$$g(n) = \sum_{k=1}^{n} h(k)\ g(n-k)\ , \qquad \text{where } g(0) = 1.$$

But by Theorem (2.3.3),

$$h(k) = 1 \qquad \text{if } k = 1, 3, 4 \text{ or } 5,$$

$$h(k) = 0 \qquad \text{otherwise.}$$

Hence $g(n) = g(n-1) + g(n-3) + g(n-4) + g(n-5)$,

where $g(k) = 0$ if $k < 0$ and $g(0) = 1$. //.

The following table lists for small values of n the simple score structures of order n and compares $g(n)$ with $l(n)$, the total number of score structures of order n.

<u>(2,3,6) Table</u>

n	g(n)	l(n)	*simple score structures*
1	1	1	(0)
2	1	1	(0,1)
3	2	2	(0,1,2) (1,1,1)
4	4	4	(0,1,2,3) (0,2,2,2) (1,1,1,3) (1,1,2,2)
5	7	9	(0,1,2,3,4) (0,1,3,3,3) (0,2,2,2,4) (0,2,2,3,3) (1,1,1,3,4) (1,1,2,2,4) (2,2,2,2,2)
6	11	22	
7	18	59	

CHAPTER 3. SEMIORDERS

SECTION 3.1 Introduction

Luce [30] and others, see for example [3, pp.263-266], have questioned the traditional assumption of transitivity of the indifference relation in paired comparison experiments. A transitive indifference relation assumes perfect discrimination on the part of the judge. Armstrong [2, p.122] suggests that a judge usually has imperfect powers of discrimination, so that preferences become recognisable only when of sufficient magnitude.

For this reason we say that an oriented graph is *representable* as a consistent preference pattern if it is possible to assign a real-valued function f to the vertices of D, such that for all u and v,

$$uRv \iff f(u) \geq g(f(v)) > f(v)$$

where g, defined over a suitable domain, is any given strictly-increasing, real-valued function for which $g(x) > x$. This is a generalisation of a definition in [42] and [38]. The function f may be thought of as measuring the judge's estimate of the characteristic under consideration and $g(f(v))$ as the threshold value of $f(v)$, so that if $f(u) \geq g(f(v))$ then u will be preferred to v.

It is easily deduced (Lemma (3.2.1)) that a representable oriented graph must satisfy the axioms of a semiorder, as defined in section 1.4. In section 3.2 those oriented graphs representable as consistent preference patterns are characterised as semiorders, replacing the weak orders of perfect discrimination and transitive indifference. In section 3.3 we show that the number of semiorders with n vertices is the nth Catalan number and count labelled semiorders. In section 3.4 an application is given to significance testing in statistics. In section 3.5 it is shown that Harary's reconstruction conjecture is true for semiorders. In section 3.6 we consider for different types of paired comparison experiments ways of defining and evaluating a coefficient of consistency of a preference pattern.

SECTION 3.2 The characterisation of representable oriented graphs

(3.2.1) Lemma (Luce [30], 1956). A representable oriented graph is a semiorder.

Proof. Let D be a representable oriented graph, and let $g(x) > x$ be any strictly-increasing, real-valued function.

RI: Since $f(\upsilon) < g(f(\upsilon))$ then, from the definition of representability, $\upsilon I \upsilon$ and so RI is true.

S1: $uR\upsilon$ implies $f(u) \geqslant g(f(\upsilon))$ and ωRz implies $f(\omega) \geqslant g(f(z))$. Now either $f(u) \geqslant f(\omega)$ in which case $f(u) \geqslant g(f(z))$ and so uRz, or $f(\omega) \geqslant f(u)$ in which case $f(\omega) \geqslant g(f(\upsilon))$ and so $\omega R\upsilon$. Hence $uR\upsilon$ and ωRz imply uRz or $\omega R\upsilon$ and S1 is true.

S2: $uR\upsilon$ implies $f(u) \geqslant g(f(\upsilon))$ and $\upsilon R\omega$ implies $f(\upsilon) \geqslant g(f(\omega))$. Now either $f(\upsilon) \geqslant f(z)$ in which case $g(f(\upsilon)) \geqslant g(f(z))$, by the definition of g, so $f(u) \geqslant g(f(z))$ and uRz, or $f(z) \geqslant f(\upsilon)$ in which case $f(z) \geqslant g(f(\omega))$ and so $zR\omega$. Hence $uR\upsilon$ and $\upsilon R\omega$ imply uRz or $zR\omega$ and S2 is true.

Thus D is a semiorder. //.

In order to characterise representable oriented graphs we reformulate the definition of a semiorder.

(3.2.2) Lemma. An equivalent definition of a semiorder D is

RI. $\upsilon I \upsilon$

and S3. $s_u \geqslant s_\upsilon$ implies $G(u) \subseteq G(\upsilon)$ and $L(\upsilon) \subseteq L(u)$.

Proof.

RI and S3 imply D is a semiorder

RI is true by assumption.

S1: Let $uR\upsilon$ and ωRz.

If $s_u \geqslant s_\omega$ then $L(\omega) \subseteq L(u)$, by S3, and so uRz. Similarly, if $s_\omega \geqslant s_u$ then $L(u) \subseteq L(\omega)$ and so $\omega R\upsilon$. Hence either uRz or $\omega R\upsilon$ and S1 is true.

S2: Let $uR\upsilon$ and $\upsilon R\omega$.

If $s_\upsilon \geqslant s_z$ then $G(\upsilon) \subseteq G(z)$, by S3, and so uRz. Similarly, if $s_z \geqslant s_\upsilon$ then $L(\upsilon) \subseteq L(z)$ and so $zR\omega$. Hence either uRz or $zR\omega$ and S2 is true.

Thus D is a semiorder.

D is a semiorder implies RI and S3

Let D be a semiorder. Then RI is true by assumption.

S3: Suppose that $s_u \geqslant s_\upsilon$. If S3 is not true then there is a vertex ω with either $\upsilon R\omega$ and ωRu, or ωRu and $\omega I\upsilon$, or $\upsilon R\omega$ and $uI\omega$.

Case (1). $\upsilon R\omega$ and ωRu.

Since D is transitive, $\upsilon R\omega$ implies $s_\upsilon > s_\omega$ and ωRu implies $s_\omega > s_u$, and hence $s_\upsilon > s_u$ which contradicts the assumption that $s_u \geqslant s_\upsilon$. Thus $\upsilon R\omega$ and ωRu is not possible.

Case (2). ωRu and $\omega I\upsilon$.

Let z be any vertex. If uRz then ωRu, uRz and $\omega I\upsilon$ imply υRz, by S2. Hence $L(u) \subseteq L(\upsilon)$.

Conversely, if $zR\upsilon$ then ωRu, $zR\upsilon$ and $\omega I\upsilon$ imply zRu by S1. Hence $G(\upsilon) \subset G(u)$.

Altogether, $s_u < s_\upsilon$ contradicting the assumption that $s_u \geqslant s_\upsilon$. Thus ωRu and $\omega I\upsilon$ is not possible.

Case (3). $\upsilon R\omega$ and $uI\omega$.

Let z be any vertex. If uRz then $\upsilon R\omega$, uRz and $uI\omega$ imply υRz by

S1. Hence $L(u) \subset L(\upsilon)$.

Conversely, if $zR\upsilon$ then $zR\upsilon$, $\upsilon R\omega$ and $uI\omega$ imply zRu by S2. Hence $G(\upsilon) \subseteq G(u)$.

Altogether, $s_u < s_\upsilon$ contradicting the assumption that $s_u \geq s_\upsilon$.

Thus $\upsilon R\omega$ and $uI\omega$ is not possible.

Since neither ωRu and $\upsilon R\omega$ nor ωRu and $\omega I\upsilon$ is possible, then ωRu implies $\omega R\upsilon$ and so $G(u) \subseteq G(\upsilon)$. Since neither $\upsilon R\omega$ and ωRu nor $\upsilon R\omega$ and $uI\omega$ is possible, then $\upsilon R\omega$ implies $uR\omega$ and so $L(\upsilon) \subseteq L(u)$. Thus S3 is true. //.

Lemma (3.2.2) is reminiscent of Theorem 1 in [30], the score sequence of a semiorder inducing a weak ordering of the vertices, see Corollary (2.2.2), which preserves preferences in the sense that $s_u \geq s_\upsilon$ implies $G(u) \subseteq G(\upsilon)$ and $L(\upsilon) \subseteq L(u)$.

<u>(3.2.3) Lemma.</u> Every semiorder is representable.

<u>Proof.</u> The proof is by induction on n, the number of vertices. Suppose that every semiorder with $r < n$ vertices whose scores have been ordered is representable by a strictly-ordered function f such that $f(1) < f(2) < \ldots < f(r)$. This is vacuously true for $n = 2$.

Suppose that the semiorder D has n vertices and score sequence $s_1 \leqslant s_2 \leqslant \ldots \leqslant s_n$. Let $g(x) > x$ be any strictly-increasing, real-valued function. We show that D is representable by a function f such that $f(1) < f(2) < \ldots < f(n)$. Let vertex k have score s'_k in $D - n$ for $k = 1, 2, \ldots, n-1$. As D is a semiorder then $s'_1 \leqslant s'_2 \leqslant \ldots \leqslant s'_{n-1}$ by Lemma (3.2.2). Now $D - n$ is a semiorder and so by the induction hypothesis can be represented by a function f such that $f(1) < f(2) < \ldots < f(n-1)$. Let p, where $0 \leqslant p \leqslant n-1$, be the greatest index adjacent from n in D. Then it follows from Lemma (3.2.2) that $p + 1$ is the least index indifferent to n.

Also, $nI(p+1)$ implies that $(n-1)I(p+1)$.

If $p = 0$, then $(n-1)I1$ in $D - n$ and so $f(n-1) < g(f(1))$. Thus there exists a real number $f(n)$ such that $f(n-1) < f(n) < g(f(1))$.

If $1 \leqslant p \leqslant n - 2$, then $p + 1 \leqslant n - 1$ and so $f(p) < f(p+1)$ by assumption. Hence as g is strictly-increasing, $g(f(p)) < g(f(p+1))$. Also, $f(n-1) < g(f(p+1))$ since $(n-1)I(p+1)$. So there exists a real number $f(n)$ such that $\max\{f(n-1), g(f(p))\} < f(n) < g(f(p+1))$.

If $p = n - 1$, then $nR(n-1)$ in D, and we choose $f(n) \geqslant g(f(n-1)) > f(n-1)$.

Hence in all cases D is representable by a function f such that $f(1) < f(2) < \ldots < f(n)$ and Lemma (3.2.3) follows by induction. //.

From Lemmas (3.2.1) and (3.2.3) we deduce the following characterisation of representable oriented graphs.

<u>(3.2.4) Theorem.</u> An oriented graph is representable if and only if it is a semiorder. //.

Theorem (3.2.4) remains true if $\geqslant$ is replaced by $>$ in the definition of representability. Scott and Suppes [42] first proved Theorem (3.2.4) for a less general definition of representability, namely

$$uRv \iff f(u) - f(v) > 1.$$

This is the case $g(x) = x + 1$.

<u>(3.2.5) Example.</u> When measuring a set of objects for some characteristic f we take account of the imperfect discrimination of measuring instruments. If $f(v)$ is the true value for object v and $\hat{f}(v)$ is the experimental value, then allowance is made for experimental error by merely asserting that

$$\hat{f}(v) - e(\hat{f}(v)) \leqslant f(v) \leqslant \hat{f}(v) + e(\hat{f}(v))$$

where $e(\hat{f}(v))$ is the estimated (positive) maximum error.

Then $f(u)$ is measurably greater than $f(v)$ if

$$\hat{f}(u) - e(\hat{f}(u)) > \hat{f}(v) + e(\hat{f}(v)),$$

and the preference relation "is measurably greater than" is defined by

$$uRv \Longleftrightarrow \hat{f}(u) - e(\hat{f}(u)) > \hat{f}(v) + e(\hat{f}(v)).$$

So $g(x) = \inf \{y \mid y - e(y) > x + e(x)\}$, and if e is a non-decreasing function then g is a strictly-increasing function. Hence the induced preference pattern is a semiorder.

A consequence of Lemma (3.2.2) is:

(3.2.6) Corollary. If D is a semiorder with score sequence S, then $D = \tilde{D}(S)$. //.

Thus no two semiorders have the same score sequence, and so a semiorder is completely determined by its score sequence.

The score sequence S is *representable* if there is a representable oriented graph with score sequence S, that is if $\tilde{D}(S)$ is a semiorder. We now give a constructive criterion for S to be representable.

(3.2.7) Proposition. $S = (s_1, s_2, \ldots, s_n)$ is representable if and only if

(1) $\quad s_{n-x-1} < s_{n-x}$, where $x = 2n - 2 - s_n$

and (2) $\quad S' = (s_1, \ldots, s_{n-x-1}, s_{n-x} - 1, \ldots, s_{n-1} - 1)$ is representable.

Proof. Suppose that S is representable, so $\tilde{D}(S)$ is a semiorder. Then it follows from Lemma (3.2.2) that nRk if $1 \leq k \leq n - x - 1$ and nIk if $n - x \leq k \leq n - 1$. Hence the semiorder D - n has score sequence S' as defined above and so S' is representable. Also, since nR(n-x-1) and nI(n-x) then $s_{n-x-1} < s_{n-x}$, for if $s_{n-x-1} = s_{n-x}$ there is a contradiction of S3. Thus if S is representable then (1) and (2) are true.

Conversely, suppose that (1) and (2) are true. Then $\tilde{D}(S')$ is a semiorder. Form D by adding vertex n to $\tilde{D}(S')$ with $L(n) = \{1, \ldots, n-x-1\}$

and $G(n) = \phi$. Then D has score sequence $S = (s_1, s_2, \ldots, s_n)$ as defined above, where $s_1 \leqslant \ldots \leqslant s_{n-x-1} < s_{n-x} \leqslant \ldots \leqslant s_{n-1} \leqslant s_n$ since $s'_1 \leqslant \ldots \leqslant s'_{n-x-1} \leqslant s'_{n-x} \leqslant \ldots \leqslant s'_{n-1}$, and $D = \tilde{D}(S)$.

It remains to prove that S3 is true for D. By assumption, $D - n = \tilde{D}(S')$ is a semiorder, so S3 is true for $D - n$. Hence, as $s_1 \leqslant s_2 \leqslant \ldots \leqslant s_{n-1} \leqslant s_n$, it is sufficient to show that $G(n) \subseteq G(n-1)$ and $L(n-1) \subseteq L(n)$ in D. Since $G(n) = \phi$, then $G(n) \subseteq G(n-1)$.

If $nR(n-1)$ in D then $L(n) = \{1, 2, \ldots, n-1\}$, by definition, and $L(n-1) \subset L(n)$. Alternatively, let $nI(n-1)$ in D and suppose there is a vertex $v \geqslant n - x$ with $(n-1)Rv$ in $D - n$. Then it follows from S3 for $D - n$ that $\{1, \ldots, v\} \subseteq L(n-1)$ in $D - n$. But $G(n-1) = \phi$ in $D - n$. Hence $s'_{n-1} \geqslant n - 2 + v \geqslant n - 2 + n - x = 2n - 2 - x = s_n$. Thus $s'_{n-1} \geqslant s_n$ and so $s_{n-1} = s'_{n-1} + 1 > s_n$ which contradicts $s_{n-1} \leqslant s_n$. This implies that there is no vertex $v \geqslant n - x$ with $(n-1)Rv$, and so $L(n-1) \subseteq L(n)$. Altogether, $L(n-1) \subseteq L(n)$ in D.
So S3 is true for $D = \tilde{D}(S)$ which is therefore a semiorder, and S is representable. //.
For $n \leqslant 6$, the representable score sequences of order n are indicated in Appendix 4.

Proposition (3.2.7) bears a certain resemblance to the constructive criterion of Havel [20] and Hakimi [14] for a partition of an even number to be the vertex-degree sequence of some graph (see, for example, [18, p.58]).

SECTION 3.3 Counting Semiorders

In this section it is shown that the number of semiorders with n vertices is the nth Catalan number. Suppose that $0 \leqslant s_1 \leqslant s_2 \leqslant \ldots \leqslant s_n$ in the semiorder D. Define $x_k \leqslant k$ to be the least-indexed vertex indifferent to k.

(3.3.1) Lemma. The integer sequence $(x_1, \ldots, x_n)$ defines a semiorder if and only if $x_1 = 1$, and $x_{k-1} \leqslant x_k \leqslant k$ for $k = 2, \ldots, n$.
Proof. Let D be a semiorder. By definition, $x_k \leqslant k$ for all k, and S3 implies that $x_{k-1} \leqslant x_k$ for $k = 2, \ldots, n$.
Hence $x_1 = 1$ and $x_{k-1} \leqslant x_k \leqslant k$ for $k = 2, \ldots, n$.

Conversely, given a sequence $(x_1, \ldots, x_n)$ satisfying the above conditions, define an oriented graph D by $L(k) = \{1, \ldots, x_k - 1\}$ for all k. Then x_k is the least-indexed vertex indifferent to k, as required, and it follows from S3 that D is a semiorder. //.

Thus the sequence $(x_1, \ldots, x_n)$ completely determines a semiorder, and it provides a simpler numerical definition of a semiorder than the score sequence. To count semiorders it is necessary to count the admissible sequences $(x_1, \ldots, x_n)$.
(3.3.2) Lemma. The number of integer sequences $(x_1, \ldots, x_n)$ for which $x_1 = 1$ and $x_{k-1} \leqslant x_k \leqslant k$ for $k = 2, \ldots, n$ is
$\frac{1}{n+1}\binom{2n}{n}$.
Proof. Let d(n) be the number of admissible sequences $(x_1, \ldots, x_n)$ of order n, and E(n, r) the subset of these sequences for which r is the greatest index k such that $x_k = k$. The value r is well defined because

$x_1 = 1$, and so $d(n) = \sum_{r=1}^{n} |E(n, r)|$.

Consider a sequence $(x_1, \ldots, x_{r-1}, r, x_{r+1}, \ldots, x_n)$ of $E(n, r)$. Then $(x_1, \ldots, x_{r-1})$ satisfies the conditions $x_1 = 1$ and $x_{k-1} \leqslant x_k \leqslant k$ for $1 \leqslant k \leqslant r - 1$. Hence the number of admissible sequences $(x_1, \ldots, x_{r-1})$ is $d(r-1)$.

Also, $(x_{r+1}, \ldots, x_n)$ satisfies the condition $x_{k-1} \leqslant x_k < k$ for $r + 1 \leqslant k \leqslant n$. Hence $x_r = r$ implies that $x_{r+1} = r$.
Now consider the sequence $(m_1, \ldots, m_{n-r})$ where $m_{k-r} = x_k - (r-1)$ for $r + 1 \leqslant k \leqslant n$. Then $x_{r+1} = r$ and $x_{k-1} \leqslant x_k < k$ for $r + 1 < k \leqslant n$ imply that $m_1 = 1$ and $m_{k-1} \leqslant m_k \leqslant k$ for $2 \leqslant k \leqslant n - r$. Hence the number of sequences $(m_1, \ldots, m_{n-r})$ is $d(n-r)$, and so there are $d(n-r)$ sequences $(x_{r+1}, \ldots, x_n)$ satisfying the required conditions.

Thus $|E(n, r)| = d(r-1)\, d(n-r)$. Hence $d(n) = \sum_{r=1}^{n} |E(n, r)| = \sum_{r=1}^{n} d(r-1)\, d(n-r)$, where $d(0) = 1$. This recurrence may be solved using standard methods, see [15, pp.25-26], to give $d(n) = \frac{1}{n+1}\binom{2n}{n}$. //.

$\frac{1}{n+1}\binom{2n}{n}$ is the nth *Catalan number*. It is the number of different ways of combining the (n+1)-sequence $a_1 a_2 \ldots a_{n+1}$ in that order using a non-associative binary product, see [15, pp.25-26]. A detailed account of these numbers is contained in [1]. Lemma (3.3.2) may also be proved by describing a one-to-one correspondence between the sequences $(x_1, \ldots, x_n)$ defined in Lemma (3.3.2) and the different ways of combining the sequence $a_1 \ldots a_{n+1}$ in the manner specified above. Suppose that, starting from the left end of the sequence $a_1 \ldots a_{n+1}$, the kth left bracket is between a_{x_k-1} and a_{x_k}. Then it can be shown that a necessary and sufficient restriction on the sequence $(x_1, \ldots, x_n)$ is that $x_1 = 1$ and $x_{k-1} \leqslant x_k \leqslant k$ for $k = 2, \ldots, n$, which is as in Lemma (3.3.2).

It follows from Lemmas (3.3.1) and (3.3.2) that:

<u>(3.3.3) Theorem.</u> The number of semiorders with n vertices is $\frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number. //.

This confirms a result of Wine and Freund in [50]. By a different method, they showed that the number of decision patterns between n objects is $\frac{1}{n+1}\binom{2n}{n}$, without giving a formal definition of a decision pattern. It was inferred, see [12], that they had counted acyclic digraphs, but it is implicit in [50] and an earlier paper by Duncan [11] that decision patterns are semiorders, as will be shown in section 3.4.

We now turn to counting the number of non-isomorphic labelled semiorders with n vertices. An *automorphism* of a labelled digraph D is an isomorphism of D with itself, that is a *permutation* of the vertex set $\{1, 2, \ldots, n\}$ which preserves adjacency. The vertices u and υ of D are *similar* if there is an automorphism of D which maps u to υ. By S3, u and υ are similar in the semiorder D if and only if $s_u = s_\upsilon$. Therefore the number of labelled semiorders with n vertices is the number of labelled representable score sequences of order n.

Define $y_k \geqslant k$ to be the greatest-indexed vertex indifferent to k in the semiorder D. Then $y_k = \max\{\upsilon \mid x_\upsilon \leqslant k\}$. The number of non-isomorphic labellings of the semiorder D is the number of different labellings of the double sequence $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$, since u and υ are similar in D if and only if $s_u = s_\upsilon$, that is if $x_u = x_\upsilon$ and $y_u = y_\upsilon$.

Let ld(n) be the number of labelled semiorders with n vertices and lc(n, r) the number of labelled semiorders for which $x_n = r$, that is $s_n = n + r - 2$. Then $1 \leqslant r \leqslant n$, so $ld(n) = \sum_{r=1}^{n} lc(n, r)$. To evaluate lc(n, r) we systematically determine all the admissible sequences $(x_1, \ldots, x_{n-1}, r)$, and then for each sequence calculate the number of labellings of the double sequence $\{(x_1, y_1), \ldots, (x_n, y_n)\}$. The results of a computer enumeration of ld(n) and lc(n, r) for $n \leqslant 11$ and $1 \leqslant r \leqslant n$ are displayed in Appendix 2.

Wine and Freund [50] asked for the total number of distinct decision sets in the pairwise comparison of n objects. Since a decision set is just a labelled semiorder, see section 3.4, there are ld(n) decision sets for n objects.

Let c(n, r) be the number of semiorders with n vertices for which $x_n = r$.

<u>(3.3.4) Proposition.</u> $c(n, r) = c(n-1, r) + c(n, r-1)$, where $c(n, r) = 0$ unless $1 \leqslant r \leqslant n$, and $c(n, 1) = 1$.

<u>Proof.</u> Suppose a semiorder has $x_n = r$. Then either $x_{n-1} = r$ or $x_{n-1} \leqslant r - 1$. But the number of semiorders with $x_n = r$ and $x_{n-1} = r$ is $c(n-1, r)$, and the number of semiorders with $x_n = r$ and $x_{n-1} \leqslant r - 1$ is $c(n, r-1)$. Hence $c(n, r) = c(n-1, r) + c(n, r-1)$. Now $c(n, r) = 0$ unless $1 \leqslant r \leqslant n$, since $1 \leqslant x_n \leqslant n$. Further, $c(n, 1) = 1$ since $x_n = 1$ implies $x_1 = \ldots = x_{n-1} = 1$ also. //.

<u>SECTION 3.4 An application to multiple comparison tests</u>

In statistics we can inquire whether the means of n given populations are all equal. This may be accomplished using an F-test in an analysis of variance. However, rejection of the hypothesis of homogeneity gives no decisions as to exactly which differences between sample means may be accounted significant, a problem considered by Duncan [11]. We wish to know which differences are significant at the chosen level.

Suppose we have n samples of the same size, with sample means $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n$, assumed to be drawn independently from normal populations with unknown means $\mu_1, \mu_2, \ldots, \mu_n$, respectively, and common unknown standard deviation σ. Following [11], the statement "$\overline{x}_u$ is significantly greater than $\overline{x}_\upsilon$", written $uR\upsilon$, is equivalent to acceptance of the decision that $\mu_u > \mu_\upsilon$; and the statement "$\overline{x}_u$ and $\overline{x}_\upsilon$ are not significantly different", written $uI\upsilon$ or υIu, is equivalent to claiming that there is insufficient evidence to decide whether μ_u is less than, equal to or greater than μ_υ, so μ_u is *unranked* relative to μ_υ.

Several *multiple comparison* test procedures for the systematic investigation of differences between sample means are examined by Duncan (see also [13, chapter II-1] and [9, pp.38-40]). In the α% level *least significant difference test*, applied after a significant result of an F-test,

$$uR\upsilon \Longleftrightarrow \overline{x}_u - \overline{x}_\upsilon > \sqrt{2}\, t_{\alpha/2}\, s$$

where s is the standard error of the mean and $t_{\alpha/2}$ is the upper $\alpha/2$ percentage point of the t distribution with the appropriate degrees of freedom. From the definition of representability, with $g(x) = x + \sqrt{2}\, t_{\alpha/2}\, s$, and Theorem (3.2.4) we conclude that the *decision patterns* induced by this test are just semiorders and that *decision sets* (see [11], [50]) are labelled semiorders. The same is true for the *multiple range tests* and *new multiple range test*. Indeed, if its conclusions are not to appear anomalous, any multiple comparison test should exhibit this property.

SECTION 3.5 On the reconstruction of semiorders

The problem of specifying a complete set of invariants of a digraph or graph is related to the following conjecture of Ulam [49, p.29].

(3.5.1) Ulam's Conjecture. Any graph G with at least three vertices is uniquely determined up to isomorphism by its *collection* of vertex-deleted subgraphs $G_\upsilon = G - \upsilon, \upsilon = 1, \ldots, n$.

Harary [16] reinterpreted the problem as one of reconstructing any graph G from its given collection of subgraphs G_υ. This can be done uniquely if and only if the conjecture is true.

A stronger form of Ulam's Conjecture is proposed in [16, Problem IV].

(3.5.2) Harary's Conjecture. Any graph G with at least four vertices can be reconstructed uniquely from its *set* of non-isomorphic vertex-deleted subgraphs G_υ, $\upsilon = 1, \ldots, r \leqslant n$.

No counterexample to either conjecture is known. The equivalent versions of these conjectures for digraphs are not in general true, see [31, p.208] for counterexamples with five and six vertices, but certain classes of digraphs can be reconstructed.

The purpose of this section is to prove that Harary's Conjecture is true for semiorders:

(3.5.3) Theorem. Any semiorder D with at least four vertices can be reconstructed uniquely from its set of non-isomorphic vertex-deleted subgraphs $D_\upsilon = D - \upsilon$.

Before a proof of Theorem (3.5.3) can be given, some preliminary results are required.

(3.5.4) Lemma. Any partial order with four vertices can be reconstructed from its set of non-isomorphic subgraphs D_υ.

Proof. A digraph D with at least four vertices is a partial order if and only if every subgraph D_υ is a partial order. Hence Lemma (3.5.4) can be verified by inspection of the vertex-deleted subgraphs of the sixteen partial orders with four vertices (see Appendix 4). //.

The next result is easily deduced.

(3.5.5) Lemma. Let D be a digraph with at least five vertices. Then D is a semiorder if and only if every subgraph D_υ is a semiorder. //.

Manvel [32, Theorem 3] has shown that the degree sequence $\{(|L(\upsilon)|,|G(\upsilon)|)\}$ of any oriented graph with at least five vertices can be derived from its collection of vertex-deleted subgraphs, and hence its score sequence can also be derived. In view of Lemmas (3.5.4) and (3.5.5) and the fact that a semiorder is completely determined by its score sequence, we can therefore assert that a semiorder with at least four vertices can be reconstructed from its collection of vertex-deleted subgraphs.

The following result is also due to Manvel [31, Proposition 2].

(3.5.6) Lemma. The number of edges of a graph G with at least four vertices can be determined from its set of non-isomorphic subgraphs G_υ. //.

Lemma (3.5.6) has an analogue for oriented graphs:

(3.5.7) Corollary. The number of arcs of an oriented graph D with at least four vertices can be determined from its set of non-isomorphic subgraphs D_υ.

Proof. Ignoring the orientations of the arcs of D and its subgraphs D_υ, we have a graph G and subgraphs G_υ. Now the set $\{G_\upsilon\}$ certainly includes all the non-isomorphic vertex-deleted subgraphs of G since $\{D_\upsilon\}$ is the set of all the non-isomorphic vertex-deleted subgraphs of D. Hence according to Lemma (3.5.6) we can determine the number of edges of G, which is just the number of arcs of D. //.

A *transmitter* is a vertex whose indegree is zero (this definition differs slightly from that in [17]). Every partial order has at least one transmitter - any vertex of greatest score. If a partial order has only one transmitter then that vertex is a *source*, being adjacent to all other vertices.

Proof of Theorem (3.5.3). Suppose we have a list of the non-isomorphic subgraphs D_υ of a digraph D with at least four vertices. If D is a semiorder with exactly four vertices then by Lemma (3.5.4) it is reconstructible. If D has five or more vertices, then by Lemma (3.5.5) we can recognise from the set D_υ whether or not D is a semiorder. Once it is known that D is a semiorder, then by Corollary (3.5.7) we can further evaluate the number of arcs in D. Therefore suppose we have determined that D is a semiorder with $n \geqslant 5$ vertices and d arcs. It remains to reconstruct D.

Two cases will be distinguished.

(1) Every D_υ with more than one transmitter has $d - (n-1)$ arcs;

(2) There is a D_υ with more than one transmitter and more than $d - (n-1)$ arcs.

Case (1). Every D_υ with more than one transmitter has $d - (n-1)$ arcs. Then D cannot have more than two transmitters since deleting a transmitter would form a subgraph D_υ with more than one transmitter and more than $d - (n-1)$ arcs. Moreover, if D does not have a source and there is a non-transmitter υ incident with fewer than $n - 1$ arcs in D, then the subgraph D_υ has at least two transmitters and more than $d - (n-1)$ arcs. Hence,

either (A) D has a source,

or (B) D has two transmitters and every non-transmitter is adjacent to or from every other vertex.

In case (1A), either D is a linear order and so there is just one D_υ which is also a linear order (type I), or there is a D_υ which has a source but is not complete (type II). For if D is complete, it is a linear order and has only one D_υ, which is a linear order (type I). Also, since D has a source and at least five vertices, then if D is not complete it has an incomplete D_υ with a source (type II).

In case (1B) there are two D_υ. D_1 is a linear order, and D_2 has two transmitters dominating a linear order with $n - 3$ vertices.

Thus it is possible to distinguish between (1A) and (1B) on the basis of the subgraphs D_υ, and to reconstruct D in case (1B). Otherwise we can assume that D has a source, case (1A). Suppose that the r vertices with the greatest scores in D form a linear order but that the $r + 1$ vertices with the greatest scores do not (so $2 \leqslant r \leqslant n$). If $r = n$, then D is of type I, and can be reconstructed since type I is distinguishable from type II on the basis of the subgraphs D_υ.

If $r < n$, then D is of type II. In this eventuality there are some D_υ whose r vertices with the greatest scores form a linear order (when the deleted vertex is not one of the r vertices with the greatest scores in D), whereas in the remaining D_υ only the $r - 1$ vertices with the greatest scores form a linear order (when the deleted vertex is one of the r vertices with the greatest scores in D). So D can be reconstructed by choosing one of the latter subgraphs D_υ and adding a source υ adjacent to every vertex of D_υ.
This completes the proof of case (1).

Case (2). There is a D_υ with more than one transmitter and more than $d - (n-1)$ arcs.

Then D does not have a source. For if D has a source u then D_u has $d - (n-1)$ arcs and any other subgraph D_υ has u as a source.

So let the semiorder D, with $s_1 \leqslant s_2 \leqslant \ldots \leqslant s_n$, have $r \geqslant 2$ transmitters, namely $n - r + 1, \ldots, n$. Then

for $\upsilon = 1, \ldots, n - r$, D_υ has r transmitters,

for $\upsilon = n - r + 1, \ldots, n - 1$, D_υ has $r - 1$ transmitters,

and D_n has at least $r - 1$ transmitters.

Since $r \geqslant 2$, the set $\{n - r + 1, \ldots, n - 1\}$ is not empty.

Hence the minimum number of transmitters in any D_υ is $r - 1$, and the deleted vertex is then a transmitter. Now choose a subgraph D_υ with a minimum number of transmitters. Let D_υ have d_υ arcs. As D is a semiorder, it can be reconstructed by adding a transmitter υ adjacent to the $d - d_\upsilon$ vertices $\{1, \ldots, d - d_\upsilon\}$ with the least scores in D_υ.

This completes the proof of Theorem (3.5.3). //.

The investigation of Ulam's and Harary's Conjectures can be widened from semiorders to all partial orders with at least four vertices. As an extension of Lemma (3.5.4), the author has verified that Harary's Conjecture, and hence also Ulam's Conjecture, is true for partial orders with five and six vertices (see Appendix 4 for diagrams of all these partial orders). Now recall that the degree sequence of an oriented graph with at least five vertices can be derived from its collection of subgraphs D_υ (see Theorem 3 of [32]). Hence a necessary condition for a pair of partial orders to be a counterexample to Ulam's Conjecture as well as Harary's Conjecture is that both should have the same degree sequence and at least seven vertices.

SECTION 3.6 Coefficients of consistency in paired comparison experiments

In sections 2.1, 3.1 and 3.2 we considered the notions of consistency in different types of paired comparison experiments:

(1) if indifference is not permitted, then a consistent preference pattern is one for which preference is transitive - a linear order;

(2) if indifference is permitted and the judge's discrimination assumed to be perfect, then a consistent preference pattern is one for which both preference and indifference are transitive - a weak order;

(3) if indifference is permitted but the judge's preferences become recognisable only when of sufficient magnitude, then a consistent preference pattern is *representable* - a semiorder.

Since judges are seldom completely consistent, it is useful to have a statistical measure of the degree of consistency among the comparisons. Such a measure was proposed for experiments of the first type by Kendall and Babington Smith [21]. In these experiments, transitive triples indicate consistent sets of choices and cyclic triples indicate inconsistent sets of choices. Kendall and Babington Smith defined the *coefficient of consistency (consistence)* ζ of a tournament T with $n \geq 3$ vertices by

$$\zeta(T) = 1 - c(T) / c_{max}(n)$$

where $c(T)$ is the number of cyclic triples in T and $c_{max}(n)$ is the maximum possible number of cyclic triples in a tournament with n vertices. The coefficient is normalised so that $0 \leq \zeta \leq 1$, with $\zeta = 1$ if and only if T is a linear order and $\zeta = 0$ if and only if T has the maximum possible number of cyclic triples.

The following results (3.6.1) to (3.6.5) from [21], [27] and [17] are restated for convenience.

(3.6.1) Lemma [17, Theorem 11.10]. The number t of transitive triples in a tournament T with score structure $(l_1, l_2, \ldots, l_n)$ is $t(T) = \sum_{i=1}^{n} \binom{l_i}{2}$.

Proof. The number of transitive triples in T with i as the source is $\binom{l_i}{2}$, since each choice of two vertices dominated by i defines exactly one transitive triple. //.

(3.6.2) Corollary [17, Corollary 11.10b]. $c(T) = \binom{n}{3} - \sum_{i=1}^{n} \binom{l_i}{2}$.

Proof. $c(T) + t(T) = \binom{n}{3}$. //.

This shows that two tournaments with the same score structure have the same number of cyclic triples, and hence the same coefficient of consistency.

(3.6.3) Lemma [21].

$$c_{max}(n) = \frac{1}{24}(n^3 - n) \text{ if } n \text{ is odd, and}$$

$$= \frac{1}{24}(n^3 - 4n) \text{ if } n \text{ is even.}$$

Proof. From Corollary (3.6.2), $c(T)$ is maximised when $\sum_{i=1}^{n} \binom{l_i}{2}$ is minimised. Now

$$\sum_{i=1}^{n} \binom{l_i}{2} = \frac{1}{2} \sum_{i=1}^{n} l_i^2 - \frac{1}{2} \cdot \frac{1}{2} n(n-1),$$

since $\sum_{i=1}^{n} l_i = \binom{n}{2}$.

So $\sum_{i=1}^{n} \binom{l_i}{2}$ is minimised when $\sum_{i=1}^{n} l_i^2$ is minimised. But $\sum_{i=1}^{n} l_i^2$ is minimised when the scores are as nearly equal as possible: if n is odd this is when $l_i = \frac{1}{2}(n-1)$ for all i, and so the minimum value of $\sum_{i=1}^{n} l_i^2$ is $\frac{1}{4}n(n-1)^2$; if n is even it is when

$$l_1 = \ldots = l_{\frac{n}{2}} = \frac{n}{2} - 1,$$

and $$l_{\frac{n}{2}+1} = \ldots = l_n = \frac{n}{2},$$

and the minimum value of $\sum_{i=1}^{n} l_i^2$ is $\frac{1}{4}n(n^2 - 2n + 2)$.

The minimum value of $\sum_{i=1}^{n} \binom{l_i}{2}$ can now be evaluated and substituted into Corollary (3.6.2) to yield $c_{max}(n)$ as stated. //.

On substituting the values of $c_{max}(n)$ into the definition of ζ we obtain the following result.

(3.6.4) Theorem [21]. For $n \geq 3$,

$$\zeta(T) = 1 - \frac{24c(T)}{n^3 - n} \quad \text{if } n \text{ is odd, and}$$

$$= 1 - \frac{24c(T)}{n^3 - 4n} \quad \text{if } n \text{ is even.} \qquad //.$$

$c(T)$ can be evaluated directly or by using Corollary (3.6.2).

$s^2 = \frac{1}{n}\sum_{i=1}^{n} (l_i - \frac{n-1}{2})^2$ is the *variance* of the scores $(l_1, l_2, \ldots, l_n)$.

The following result [9, p.22] is implicit in [21].

(3.6.5) Lemma. $ns^2 = \frac{1}{12}n(n^2 - 1) - 2c(T)$.

Proof. $ns^2 = \sum_{i=1}^{n} l_i^2 - \frac{1}{4}n(n-1)^2$ (using the alternative formula for the variance s^2)

$$= 2\sum_{i=1}^{n} \binom{l_i}{2} + \sum_{i=1}^{n} l_i - \frac{1}{4}n(n-1)^2$$

$$= 2\{\binom{n}{3} - c(T)\} + \binom{n}{2} - \frac{1}{4}n(n-1)^2 \text{ (from Corollary (3.6.2))}$$

$$= \frac{1}{12}n(n^2 - 1) - 2c(T). \qquad //.$$

s^2 is a maximum when $c(T) = 0$, that is when $L = (0, 1, \ldots, n - 1)$, and the *maximum value of the variance* is $s^2_{max}(n) = \frac{1}{12}(n^2 - 1)$, see also

[21] or [27].

Landau [27] defined the *hierarchy index* h of a tournament T with score structure $(l_1, l_2, \ldots, l_n)$ by $h(T) = \frac{12}{n^3-n} \sum_{i=1}^{n} (l_i - \frac{n-1}{2})^2$.

Thus $h = s^2/s^2_{max}(n)$, and so h takes values between 0 and 1 with h = 1 if and only if T is a linear order. The hierarchy index is similar to the coefficient of consistency in that it provides a measure of how close a tournament is to being a linear order.

The relationship between h and ζ is as follows.

(3.6.6) Theorem. For $n \geqslant 3$,

$$\begin{aligned} h &= \zeta \qquad \text{if } n \text{ is odd, and} \\ &= \zeta + \frac{3}{n^2-1}(1-\zeta) \text{ if } n \text{ is even.} \end{aligned}$$

Proof. Since $s^2_{max}(n) = \frac{1}{12}(n^2 - 1)$ it follows from Lemma (3.6.5) that $s^2/s^2_{max}(n) = 1 - 24c(T)/(n^3 - n)$. Hence $h = \zeta$ if n is odd.

Now $1 - \frac{24c(T)}{n^3 - n} = 1 - \frac{24c(T)}{n^3 - 4n} + \frac{3}{n^2-1} \cdot \frac{24c(T)}{n^3 - 4n}$.

So if n is even, $h = \zeta + \frac{3}{n^2-1}(1-\zeta)$. //.

Thus h and ζ are identical for odd n. Also, if n is even then h > 0.

Of course the above definition of consistency does not extend to experiments in which indifference is permitted. However, it is possible to use the same general approach to the notion of consistency. In experiments of the second type, weakly ordered triples indicate consistent sets of choices and other triples indicate inconsistencies. For such experiments we might therefore define the *coefficient of consistency* η of an oriented graph D with $n \geqslant 3$ vertices by

$$\eta(D) = \frac{\omega(D) - \omega_{min}(n)}{\binom{n}{3} - \omega_{min}(n)}$$

where $\omega(D)$ is the number of weakly ordered triples in D and $\omega_{min}(n)$ is the minimum possible number of weakly ordered triples in an oriented graph with n vertices. The coefficient is normalised so that $0 \leqslant \eta \leqslant 1$, and $\eta = 1$ if and only if D is a weak order.

$\omega(D)$ is easily evaluated, although unlike $t(T)$ it is not determined by the score sequence of D. The triples  and  have the same score sequence but whereas the first is a weak order, the second is not.

(3.6.7) Proposition. $\omega_{min}(n) = 0$ for $3 \leqslant n \leqslant 6$.

Proof. The oriented graph  has six vertices and no weakly ordered triples. So $\omega_{min}(6) = 0$. Moreover, its induced subgraphs have no weakly ordered triples so $\omega_{min}(n) = 0$ for $n < 6$. //.

(3.6.8) Corollary. If D is an oriented graph with $3 \leqslant n \leqslant 6$ vertices, then $\eta(D) = \omega(D) / \binom{n}{3}$. //.

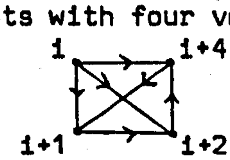
Notice that if T is a tournament then although $\omega(T) = t(T) = \binom{n}{3} - c(T)$, the coefficient $\eta(T)$ will differ from $\zeta(T)$ for $n \geqslant 4$ because $\omega_{min}(n)$ differs from $\binom{n}{3} - c_{max}(n)$. For example, $\omega_{min}(4) = 0$ while $c_{max}(4) = 2$.

In experiments of the third type, consistent preference patterns are semiorders. A semiorder is defined in terms of forbidden induced subgraphs with four vertices - axioms S1 and S2. An induced 4-vertex subgraph of a preference pattern indicates a consistent set of choices if the subgraph is a semiorder; otherwise the subgraph indicates an inconsistent set of choices. Accordingly, for such experiments we define the *coefficient of consistency* θ of an oriented graph D with $n \geqslant 4$ vertices by

$$\theta(D) = \frac{s(D) - s_{min}(n)}{\binom{n}{4} - s_{min}(n)}$$

where s(D) is the number of 4-vertex induced subgraphs of D that are semiorders, and $s_{min}(n)$ is the minimum value of s(D) for oriented graphs with n vertices. Then $0 \leqslant \theta \leqslant 1$, with $\theta = 1$ if and only if D is a semiorder.

<u>(3.6.9) Theorem.</u> $s_{min}(n) = 0$ for $4 \leqslant n \leqslant 14$.

<u>Proof.</u> Let T_7 be the tournament with vertices {1, 2, ..., 7} in which iRj if $j \equiv i+1$, $i+2$ or $i+4$ (mod 7), and jRi otherwise. Then T_7 has no transitive subtournaments with four vertices, since the 4-vertex subtournament with source i is 

, which is not transitive.

Let the oriented graph D with 14 vertices be the *disjoint union* (see section 4.2 for definition) of two copies of T_7. The induced 4-vertex subgraphs of D are of the following types:

(a) an intransitive tournament (if all four vertices are from the same tournament T_7, and since T_7 has no transitive subtournaments of order four),

(b)  or  (if the subgraph has three vertices from one tournament T_7 and one from the other),

(c)   (if the subgraph has two vertices from each tournament).

None of these subgraphs are semiorders, so s(D) = 0. Thus $s_{min}(14) = 0$.

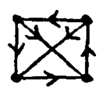
If n < 14 take any induced subgraph D' of D with n vertices. Then $s(D') = 0$, and so $s_{min}(n) = 0$ for n < 14. //.

<u>(3.6.10) Corollary</u>. If D is an oriented graph with $4 \leq n \leq 14$ vertices, then $\theta(D) = s(D) / \binom{n}{4}$. //.

So for paired comparison experiments with up to 14 objects, θ can be evaluated as the proportion of (induced) 4-vertex subgraphs that are semiorders.

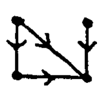
In general $\theta(D) \neq \eta(D)$, as the following examples with four vertices show.

$$c_{max}(4) = 2, \ \omega_{min}(4) = 0, \ s_{min}(4) = 0.$$



$c = \omega = 2$, $s = 0$,

$\zeta = 0$, $h = 1/5$, $\eta = 1/2$, $\theta = 0$.



$\omega = 3$, $s = 1$,

$\eta = 3/4$, $\theta = 1$.

The difficulties in computing the measures η and θ for arbitrary oriented graphs lie in determining $\omega_{min}(n)$ and $s_{min}(n)$.

CHAPTER 4 FINITE TOPOLOGIES AND TRANSITIVE DIGRAPHS

SECTION 4.1 Introductory definitions and results

A *finite topology* (V, T) is a finite non-empty set V of *points* together with a family T of *open* subsets of V, such that the union and intersection of two open sets are open, as are the empty set ϕ and V itself. $|V|$ is the *order* of (V, T). All topologies in this chapter are finite. A *labelled topology* has the point set $\{1, 2, \ldots, n\}$. Two topologies are *homeomorphic* if there is a one-to-one correspondence between their point sets which preserves open sets. A "topology" is taken to mean a homeomorphism class of topologies. These definitions correspond closely to those of [12].

The following fundamental theorem indicates a close relationship between topologies and transitive digraphs.

(4.1.1) Theorem [43], [24], [12]. There is a one-to-one correspondence between the labelled topologies with n points and the labelled transitive digraphs with n vertices.

Before a proof of Theorem (4.1.1) is given, it is convenient to describe the correspondence and introduce some further definitions.

If $u, \upsilon \in V$ and u is a member of every open set of T containing υ, then υ is *inseparable* from u in T. Otherwise υ is *separable* from u.

The correspondence. A digraph $D(T)$ is associated with T as follows: the vertex set of $D(T)$ is V. For two vertices u and υ of V, $uR\upsilon$ in $D(T)$ if and only if υ is inseparable from u in T.

Under this correspondence $\upsilon R\upsilon$ so $D(T)$ is reflexive, although digraphs were originally defined to be irreflexive.

The *basic open set* $B(\upsilon)$ of υ in T is the minimal open set containing υ. Then $\upsilon \in B(\upsilon)$ and $B(\upsilon) = G(\upsilon)$ where $G(\upsilon)$ is the set of vertices adjacent to υ in $D(T)$. A non-empty subset of V is an open set of T if and only if it is a union of basic open sets. For any non-empty subset V_1 of V,

$$B(V_1) = \bigcup_{\upsilon \varepsilon V_1} B(\upsilon)$$

is the minimal open set containing V_1.

Proof of Theorem (4.1.1). Clearly, $D(\mathcal{T})$ is transitive and is uniquely determined by $(V, \mathcal{T})$.

We next show that to each labelled transitive digraph D with vertex set $V = \{1, 2, \ldots, n\}$ there corresponds a unique labelled topology. Define $\mathcal{T}(D)$ to be that labelled topology with point set V in which the basic open sets are all sets $G(\upsilon)$, $\upsilon \varepsilon V$. Then every non-empty open set of $\mathcal{T}(D)$ is of the form $\bigcup_{\upsilon \varepsilon V_1} G(\upsilon)$ for some subset V_1 of V. We must show that $\mathcal{T}(D)$ is indeed a topology. By definition, the union of two open sets is open. To prove that the intersection of two open sets is open, it is sufficient to show that the intersection of two basic open sets $G(u)$ and $G(\upsilon)$ is open. Now $G(u) \cap G(\upsilon)$ is the set of vertices adjacent to both u and υ. As D is transitive, then

$$G(u) \cap G(\upsilon) = \bigcup_{x \varepsilon G(u) \cap G(\upsilon)} G(x).$$

Hence $G(u) \cap G(\upsilon)$ is an open set, being a union of basic open sets.

Finally, the correspondence is one-to-one since it follows directly from the definitions of $D(\mathcal{T})$ and $\mathcal{T}(D)$ that $D(\mathcal{T}(D)) = D$. //.

The above proof is taken from [12]. In normal usage graphs and digraphs are defined to be irreflexive. Therefore it is convenient to think of $D(\mathcal{T})$ as being irreflexive, it being understood that a point is inseparable from itself.

Theorem (4.1.1) remains valid if the word "labelled" is omitted, being interpreted as a one-to-one correspondence between (non-homeomorphic) topologies of order n and transitive digraphs of order n. $(V, \mathcal{T}_1)$ and $(V, \mathcal{T}_2)$ are homeomorphic if and only if $D(\mathcal{T}_1)$ and $D(\mathcal{T}_2)$ are isomorphic.

This chapter is devoted to some implications of Theorem (4.1.1) and aims to develop the natural link between topologies and transitive digraphs. Studies of the combinatorial aspects of topologies can usually be simplified by consideration of their associated digraphs. In sections 4.2 and 4.3 graphical analogues of various topological properties are presented, including connectivity, maximal connectivity and different types of unary and binary topological operations. Then in section 4.3 a solution is offered to the problem of enumerating maximal connected topologies of order n. Finally, in section 4.4 the following question is considered: given n, for which values of r is there a topology of order n with r open sets? Certain general criteria are established, enabling a solution to be provided for $n \leqslant 9$.

To begin with, some basic definitions and results are given.

<u>The T_0 (Kolmogorov) separation axiom.</u> $(V, \mathcal{T})$ is T_0 if for distinct $u, \upsilon \in V$ either u is separable from υ or υ is separable from u.

A direct consequence of this definition is:

<u>(4.1.2) Proposition</u> [6, p.14]. $(V, \mathcal{T})$ is T_0 if and only if $D(\mathcal{T})$ is a partial order. //.

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on V, $\mathcal{T}_1 \subseteq \mathcal{T}_2$ means that every open set of $\mathcal{T}_1$ is an open set of $\mathcal{T}_2$. Then unless $\mathcal{T}_1$ and $\mathcal{T}_2$ are identical, $\mathcal{T}_2$ is *finer* than $\mathcal{T}_1$, and $\mathcal{T}_1$ is *coarser* than $\mathcal{T}_2$, written $\mathcal{T}_1 \subset \mathcal{T}_2$. Similarly, $D_1 \subseteq D_2$ indicates that D_1 is a subgraph of D_2, and $D_1 \subset D_2$ means that D_1 is a *proper* subgraph of D_2.

(4.1.3) Theorem. $T_1 \subseteq T_2$ if and only if $D(T_2)$ is a spanning subgraph of $D(T_1)$.

Proof. Suppose $T_1 \subseteq T_2$. $D(T_1)$ and $D(T_2)$ have vertex set V. If uRv in $D(T_2)$ so that v is inseparable from u in T_2 then v must be inseparable from u in T_1, and so uRv in $D(T_1)$. Hence $T_1 \subseteq T_2$ implies that $D(T_2)$ is a spanning subgraph of $D(T_1)$.

Conversely, let $D(T_2)$ be a spanning subgraph of $D(T_1)$. For all $v \varepsilon V$ let $B_1(v)$ and $B_2(v)$ be the basic open sets of v in T_1 and T_2 respectively. Then $x \varepsilon B_2(x)$ and so $B_2(B_1(v)) = \bigcup_{x \varepsilon B_1(v)} B_2(x) \supseteq B_1(v)$.

Further, if $u \varepsilon B_2(B_1(v))$ then $uR\omega$ in $D(T_2)$ for some $\omega \varepsilon B_1(v)$. But as $D(T_2) \subseteq D(T_1)$ then $uR\omega$ in $D(T_1)$. Since also ωRv in $D(T_1)$ then uRv in $D(T_1)$ and so $u \varepsilon B_1(v)$. Thus $B_2(B_1(v)) \subseteq B_1(v)$. So $B_2(B_1(v)) = B_1(v)$ and $B_1(v)$ is an open set of T_2. But since $B_1(v)$ is any basic open set of T_1, every basic open set of T_1 is an open set of T_2. Therefore every open set of T_1 is an open set of T_2 which means that $T_1 \subseteq T_2$. Thus if $D(T_2)$ is a spanning subgraph of $D(T_1)$ then $T_1 \subseteq T_2$. //.

SECTION 4.2 Operations on topologies and their associated digraphs

Five common types of topologies are described below. It is a simple matter to show that they are valid topologies and derive their corresponding digraphs. In the remainder of the section three binary operations on topologies are considered.

1. The *discrete topology* has all subsets of V as open sets and $D(T)$ is totally disconnected, that is it contains no arcs.

2. The *indiscrete topology* has as open sets ϕ and V only and $D(T)$ is

a complete symmetric digraph.

The *closed sets* of $(V, \mathcal{T})$ are the complements in V of the open sets of $\mathcal{T}$.

3. The *closed topology* $(V, \mathcal{T}_c)$ has as open sets the closed sets of $(V, \mathcal{T})$, and vice versa. $D(\mathcal{T}_c)$ is the digraph formed by reversing the orientations of the arcs of $D(\mathcal{T})$.

4. The *partition topology.* If P is a partition of V into disjoint subsets, the *partition topology* $(V, \mathcal{T}_p)$ has these subsets as its basic open sets. For distinct vertices u and υ, $uR\upsilon$ in $D(\mathcal{T}_p)$ if and only if u and υ are in the same subset of P.

5. The *deleted set topology* $(V-A, \mathcal{T}|V-A)$ is obtained from $(V, \mathcal{T})$ by deleting A, a proper subset of V, from every open set of $\mathcal{T}$ and thus forming $\mathcal{T}|V-A$, a family of subsets of V-A. If A is a singleton, $(V-A, \mathcal{T}|V-A)$ is the *deleted point topology.* $D(\mathcal{T}|V-A)$ is the subgraph of $D(\mathcal{T})$ induced by V-A.

There are several important operations on topologies which produce other topologies. Types 3 and 5 above are unary operations. Three binary operations, intersection, union and cartesian product are now defined and the associated digraphs of these composite topologies determined.
First some graphical definitions are required. Throughout this section the digraphs D_1 and D_2 have vertex sets V_1 and V_2 and arc sets R_1 and R_2 respectively.
Their *union* $D_1 \cup D_2$ has vertex set $V_1 \cup V_2$ and arc set $R_1 \cup R_2$. If $V_1 \cap V_2 = \phi$ it is their *disjoint union.*
Their *intersection* $D_1 \cap D_2$ has vertex set $V_1 \cap V_2$ and arc set $R_1 \cap R_2$.
The *transitive closure* D^t of a given digraph D is the minimal transitive

digraph containing D, and has the same vertex set as D.

6. The *intersection topology* $(V_1 \cap V_2, T_1 \cap T_2)$ of (V_1, T_1) and (V_2, T_2) has as open sets those sets common to the deleted set topologies $T_1|V_1 \cap V_2$ and $T_2|V_1 \cap V_2$. If T_1 and T_2 have the same point set V then the open sets of the intersection topology $(V, T_1 \cap T_2)$ are those sets common to T_1 and T_2.

<u>(4.2.1) Theorem.</u> The digraph of the intersection of two topologies with the same point set V is the transitive closure of the union of their digraphs,

$$D(T_1 \cap T_2) = [D(T_1) \cup D(T_2)]^t .$$

<u>Proof.</u> Clearly $D(T_1 \cap T_2)$ contains any arc in $D(T_1)$ or $D(T_2)$ and since $T_1 \cap T_2$ is a topology $D(T_1 \cap T_2)$ is transitive. Thus $D(T_1 \cap T_2) \supseteq [D(T_1) \cup D(T_2)]^t$.

Next suppose that $uRv \notin [D(T_1) \cup D(T_2)]^t$. Let $G(v)$ be the set of vertices adjacent to v in $[D(T_1) \cup D(T_2)]^t$. Then $u \notin G(v)$. Further, no vertex of $G(v)$ is adjacent from any vertex of $V-G(v)$ in $[D(T_1) \cup D(T_2)]^t$. Thus no vertex of $G(v)$ is adjacent from any vertex of $V-G(v)$ in either $D(T_1)$ or $D(T_2)$. Letting $B_1(x)$ and $B_2(x)$ denote the basic open sets of x in T_1 and T_2 respectively, it follows that

$$G(v) = \bigcup_{x \in G(v)} B_1(x) = \bigcup_{x \in G(v)} B_2(x) .$$

So $G(v)$ is an open set of both T_1 and T_2, in each case the minimal open set containing itself. Thus $G(v)$ is an open set of $T_1 \cap T_2$ and $uRv \notin D(T_1 \cap T_2)$, since $u \notin G(v)$. Hence $D(T_1 \cap T_2) \subseteq [D(T_1) \cup D(T_2)]^t$. Altogether, $D(T_1 \cap T_2) = [D(T_1) \cup D(T_2)]^t$. //.

If the point sets of the topologies are not identical, Theorem (4.2.1) remains true with the modification that $D(T_1)$ and $D(T_2)$ are restricted to the vertices common to both digraphs and so are replaced by their subgraphs induced by $V_1 \cap V_2$.

7. The *union topology* $(V_1 \cup V_2, \mathcal{T}_1 \cup \mathcal{T}_2)$ of $(V_1, \mathcal{T}_1)$ and $(V_2, \mathcal{T}_2)$ has as basis all unions and intersections of open sets of $\mathcal{T}_1$ and $\mathcal{T}_2$. If $V_1 \cap V_2 = \phi$ it is their *disjoint union topology.*

(4.2.2) Theorem. $D(\mathcal{T}_1 \cup \mathcal{T}_2)$ has vertex set $V_1 \cup V_2$ and the following structure: let u and υ be distinct vertices of $V_1 \cup V_2$. Then

(1) for $\upsilon \in V_1 \cap V_2$, $uR\upsilon$ in $D(\mathcal{T}_1 \cup \mathcal{T}_2)$ if and only if $u \in V_1 \cap V_2$ and $uR\upsilon$ in $D(\mathcal{T}_1) \cap D(\mathcal{T}_2)$

(2) for $\upsilon \in V_1 - V_1 \cap V_2$, $uR\upsilon$ in $D(\mathcal{T}_1 \cup \mathcal{T}_2)$ if and only if $u \in V_1$ and $uR\upsilon$ in $D(\mathcal{T}_1)$, and similarly for $\upsilon \in V_2 - V_1 \cap V_2$, $uR\upsilon$ in $D(\mathcal{T}_1 \cup \mathcal{T}_2)$ if and only if $u \in V_2$ and $uR\upsilon$ in $D(\mathcal{T}_2)$.

Proof. Let $B_1(\upsilon)$ and $B_2(\upsilon)$ be the basic open sets of υ in $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively. It follows immediately from the definition of the union topology that:
if $\upsilon \in V_1 \cap V_2$, then υ is inseparable from u in $\mathcal{T}_1 \cup \mathcal{T}_2$ if and only if $u \in B_1(\upsilon)$ and $u \in B_2(\upsilon)$ (so $u \in V_1 \cap V_2$); if $\upsilon \in V_1 - V_1 \cap V_2$, then υ is inseparable from u in $\mathcal{T}_1 \cup \mathcal{T}_2$ if and only if $u \in B_1(\upsilon)$ (so $u \in V_1$), and similarly if $\upsilon \in V_2 - V_1 \cap V_2$, then υ is inseparable from u in $\mathcal{T}_1 \cup \mathcal{T}_2$ if and only if $u \in B_2(\upsilon)$ (so $u \in V_2$).

This is easily translated into graphical terms. $D(\mathcal{T}_1 \cup \mathcal{T}_2)$ has vertex set $V_1 \cup V_2$. Suppose u and υ are distinct vertices of $V_1 \cup V_2$.
If $\upsilon \in V_1 \cap V_2$, then $uR\upsilon$ in $D(\mathcal{T}_1 \cup \mathcal{T}_2)$ if and only if $u \in V_1 \cap V_2$ with $uR\upsilon$ in $D(\mathcal{T}_1)$ and $uR\upsilon$ in $D(\mathcal{T}_2)$, that is with $uR\upsilon$ in $D(\mathcal{T}_1) \cap D(\mathcal{T}_2)$. This is condition (1).
If $\upsilon \in V_1 - V_1 \cap V_2$, then $uR\upsilon$ in $D(\mathcal{T}_1 \cup \mathcal{T}_2)$ if and only if $u \in V_1$ and $uR\upsilon$ in $D(\mathcal{T}_1)$, and similarly if $\upsilon \in V_2 - V_1 \cap V_2$, then $uR\upsilon$ in $D(\mathcal{T}_1 \cup \mathcal{T}_2)$ if and only if $u \in V_2$ and $uR\upsilon$ in $D(\mathcal{T}_2)$, which is condition (2). //.

The following two extreme cases of union topologies provide an interesting contrast between their associated digraphs.

(4.2.3) Corollary. The digraph of the union of two topologies with the same point set is the intersection of their digraphs,

$$D(T_1 \cup T_2) = D(T_1) \cap D(T_2) \qquad \text{if } V_1 = V_2.$$

Proof. If $V_1 = V_2$ condition (1) only of Theorem (4.2.2) applies. //.

(4.2.4) Corollary. The digraph of the disjoint union of two topologies is the (disjoint) union of their digraphs,

$$D(T_1 \cup T_2) = D(T_1) \cup D(T_2) \qquad \text{if } V_1 \cap V_2 = \phi .$$

Proof. If $V_1 \cap V_2 = \phi$ condition (2) only of Theorem (4.2.2) applies. //.

8. The *cartesian product topology* [26, p.137] $(V_1 \times V_2, T_1 \times T_2)$ of (V_1, T_1) and (V_2, T_2) has as its point set the cartesian product $V_1 \times V_2$ and as open sets all unions of cartesian products $A \times B$ where $A \epsilon T_1$ and $B \epsilon T_2$.

It is necessary to stipulate that the open sets are all the unions of cartesian products, rather than just the actual products, in order to satisfy the axioms of a topology, for in general $(A_1 \times B_1) \cup (A_2 \times B_2) \neq (A_1 \cup A_2) \times (B_1 \cup B_2)$. However the cartesian products do include all their intersections since $(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2)$.

The *cartesian product* [5, p. 377] $D_1 \times D_2$ of digraphs D_1 and D_2 has vertex set $V_1 \times V_2$ with $(u_1, u_2)R(\upsilon_1, \upsilon_2)$ in $D_1 \times D_2$ if and only if $u_1 R \upsilon_1$ in D_1 and $u_2 R \upsilon_2$ in D_2. Sabidussi [40], see also [18, p.22], defines the cartesian product differently but the next result shows Berge's definition [5] to be more natural in this context.

(4.2.5) Theorem. The digraph of the cartesian product of two topologies is the cartesian product of their digraphs,

$$D(T_1 \times T_2) = D(T_1) \times D(T_2).$$

Proof. $(u_1, u_2)R(\upsilon_1, \upsilon_2)$ in $D(T_1 \times T_2)$

$\Longleftrightarrow$ (υ_1, υ_2) is inseparable from (u_1, u_2) in $(V_1 \times V_2, \mathcal{T}_1 \times \mathcal{T}_2)$

$\Longleftrightarrow$ υ_1 is inseparable from u_1 in $(V_1, \mathcal{T}_1)$ and υ_2 is inseparable from u_2 in $(V_2, \mathcal{T}_2)$

$\Longleftrightarrow$ $u_1 R \upsilon_1$ in $D(\mathcal{T}_1)$ and $u_2 R \upsilon_2$ in $D(\mathcal{T}_2)$

$\Longleftrightarrow$ $(u_1, u_2) R (\upsilon_1, \upsilon_2)$ in $D(\mathcal{T}_1) \times D(\mathcal{T}_2)$.

Thus $D(\mathcal{T}_1 \times \mathcal{T}_2) = D(\mathcal{T}_1) \times D(\mathcal{T}_2)$. //.

SECTION 4.3 The enumeration of maximal connected topologies

The topology $(V, \mathcal{T})$ is *connected* if no proper non-empty subset of V is both an open set and a closed set. Otherwise it is *separable* (*unconnected*). $(V, \mathcal{T})$ is *maximal connected* if it is connected and each topology on V that is finer than $\mathcal{T}$ is separable. Maximal connected topologies were introduced by Thomas [48] who raised the problem of counting the number of (non-homeomorphic) maximal connected topologies with n points. In this section the problem is solved by deriving some further results based on Theorem (4.1.1) and then relating the number of maximal connected topologies to the counting series for rooted trees. These results will also be useful in section 4.4 as maximal connected topologies play an important role in the study of the number of open sets of finite topologies.

$G(\mathcal{T})$ is the "graph" formed from $D(\mathcal{T})$ by omitting the orientations of the arcs. It is not strictly a graph unless $D(\mathcal{T})$ is a partial order for otherwise it will contain multiple edges. The graph G is *connected* if for every two vertices u and υ there is a sequence of edges $u\upsilon_1, \upsilon_1\upsilon_2, \ldots, \upsilon_{n-1}\upsilon_n, \upsilon_n\upsilon$ in G. A maximal connected subgraph of G is a *component* of G. If u and υ are in the same component, the *distance* $d(u, \upsilon)$ between u and υ is the number of edges in a shortest sequence of edges joining u and υ, and $d(\upsilon, \upsilon) = 0$.

<u>(4.3.1) Lemma.</u> (V, T) is connected if and only if G(T) is connected, that is if D(T) is *weakly connected.*

<u>Proof.</u> If G(T) is not connected V can be partitioned into two non-empty subsets A and V-A such that no vertex of A is adjacent to or from any vertex of V-A in D(T). Then B(A) = A and B(V-A) = V-A so A and V-A are open sets of T. Hence A and V-A are also closed sets of T and therefore T is not connected.

Conversely, suppose that G(T) is connected and again let A be any proper subset of V. Then either

(1) some $u \epsilon A$ is adjacent to some $\upsilon \epsilon V-A$, so that υ is inseparable from u and V-A cannot be an open set of T - hence A is not a closed set,

or (2) some $\upsilon \epsilon V-A$ is adjacent to some $u \epsilon A$, so u is inseparable from υ and A is not an open set.

Thus no proper subset of V is both open and closed, and so (V, T) is connected. //.

<u>(4.3.2) Lemma.</u> If (V, T) is maximal connected then D(T) is a partial order.

<u>Proof.</u> It will be shown that if (V, T) is maximal connected D(T) can have no symmetric pairs of arcs.

Suppose (V, T) is maximal connected with D(T) containing the symmetric pair $uR\upsilon$ and υRu. By Lemma (4.3.1), D(T) is weakly connected. Let A be the maximal subset of V containing u and υ within which every pair of vertices is mutually adjacent in D(T). Since D(T) is transitive and the subgraph induced by A is complete and symmetric, then V can be partitioned into subsets A, B, C and D (of which any except A may be empty) such that

(1) every vertex of A is adjacent to every vertex of B and no

vertex of B is adjacent to any vertex of A

(2) every vertex of C is adjacent to every vertex of A and no vertex of A is adjacent to any vertex of C

(3) no vertex of A is adjacent to any vertex of D and vice versa.

By deletion of arcs the complete symmetric subgraph A can be altered to a linear order, forming a transitive, weakly connected digraph D', a proper subgraph of D(T) with corresponding topology T'. By Theorem (4.1.3), T' is strictly finer than T and, by Lemma (4.3.1), T' is connected, so T is not maximal connected. This contradiction of the initial supposition shows that if (V, T) is maximal connected D(T) has no symmetric pairs and is therefore a partial order. //.

The following immediate consequence of Lemma (4.3.2) and Proposition (4.1.2) is Theorem 1 of [48] restricted to finite topologies.

(4.3.3) Corollary. A maximal connected topology is T_0. //.

A *cycle* in a graph G is a collection of distinct vertices $v_1, v_2, \ldots, v_n$ together with the edges $v_1 v_2, v_2 v_3, \ldots, v_{n-1} v_n, v_n v_1$ (distinct). A graph without any cycle is *acyclic*. A connected, acyclic graph is a *tree*. A *rooted tree* has one vertex, its *root*, distinguished from the others.

(4.3.4) Lemma. (V, T) is maximal connected if and only if G(T) is a tree, that is D(T) is an *oriented tree*.

Proof. If G(T) is a tree (V, T) is connected, by Lemma (4.3.1). Further, if T' is strictly finer than T then, from Theorem (4.1.3), G(T') is a proper subgraph of G(T). So G(T') is not connected and it follows from Lemma (4.3.1) that T' is not connected. Hence (V, T) is maximal connected.

Conversely, let (V, T) be maximal connected. It follows from Lemma

(4.3.1) that G(T) is connected. Suppose G(T) has a cycle containing the edge uv where, without loss of generality, uRv in D(T). Then either

(1) there is no intermediate vertex $w \in V$ with uRw and wRv in D(T)

or (2) there is a vertex w with uRw and wRv in D(T). Then the edge uw is part of a cycle in G(T) and the next step is to look for an intermediate vertex between u and w in D(T).

Since D(T) is a partial order, by Lemma (4.3.2), and V is finite, repetition of (2) will eventually produce a vertex z with uRz in D(T) such that the edge uz is part of a cycle and there is no intermediate vertex between u and z in D(T). Now delete the arc uRz (which may be uRv) from D(T) to form D'. D' is a partial order, because u and z have no intermediate vertex in D(T), with corresponding topology T', say. G(T') is connected because uz is part of a cycle in G(T). But by Theorem (4.1.3), T' is strictly finer than T, and by Lemma (4.3.1), T' is connected, contradicting the assumption that T is maximal connected. Hence G(T) must be acyclic, and so G(T) is a tree. //.

A partial order is *vacuously transitive* if every vertex has indegree zero (a transmitter) or outdegree zero (a *receiver*).

<u>(4.3.5) Corollary.</u> If (V, T) is maximal connected then D(T) is vacuously transitive.

<u>Proof.</u> From Lemma (4.3.4), if (V, T) is maximal connected then D(T) is a transitively oriented tree, and so is vacuously transitive. //.

The maximal connected topologies of order n are now counted in terms of trees and rooted trees. Let t(n), r(n) and m(n) be respectively the numbers of trees, rooted trees and maximal connected topologies of order n, with generating functions

$$T(x) = \sum_{n=1}^{\infty} t(n)\, x^n, \; R(x) = \sum_{n=1}^{\infty} r(n)\, x^n \text{ and } M(x) = \sum_{n=1}^{\infty} m(n)\, x^n.$$

<u>(4.3.6) Lemma.</u> $m(1) = 1,$

$m(n) = 2t(n)$ if $n > 1$ and odd,

$m(n) = 2t(n) - r(n/2)$ if n is even.

<u>Proof.</u> By Lemma (4.3.4) and Corollary (4.3.5), $m(n)$ is the number of (vacuously) transitively oriented trees of order n. Clearly, $m(1) = 1$. Now suppose $n > 1$. Since a tree is connected, the orientation of any one edge determines the transitive orientation of the tree.

The *eccentricity* of a vertex v of a tree is the maximum value of $d(u, v)$ for all $u \in V$. A vertex of minimum eccentricity is a *central vertex*. A tree has either one or two central vertices [18, Theorem 4.2]. Thus two types of trees will be distinguished.

<u>Type (1). Trees with one central vertex</u> (monocentred). A tree can be oriented with the centre either a transmitter or a receiver, so each monocentred tree has two transitive orientations.

<u>Type (2). Trees with two central vertices</u> (bicentred). Suppose a tree T has central vertices c_1 and c_2 and that removal of the edge $c_1 c_2$ decomposes T into trees T_1 and T_2 rooted at c_1 and c_2 respectively. If the rooted trees T_1 and T_2 are not isomorphic then T has two transitive orientations according to the orientation of $c_1 c_2$. If T_1 and T_2 are isomorphic then they are both of order $n/2$ and T has just one transitive orientation. But the number of bicentred trees of order n with T_1 isomorphic to T_2 is $r(n/2)$, the number of rooted trees of order $n/2$.

Altogether, $m(n) = 2t(n)$ if n is odd,

$m(n) = 2t(n) - r(n/2)$ if n is even. //.

An alternative statement of Lemma (4.3.6) is:

<u>(4.3.7) Corollary.</u> $M(x) = 2T(x) - R(x^2) - x.$ //.

Now $T(x)$ and $R(x)$ are related by the following result of Otter [35].

(4.3.8) Lemma. $T(x) = R(x) + \frac{1}{2}R(x^2) - \frac{1}{2}R^2(x)$. //.

Combining Corollary (4.3.7) and Lemma (4.3.8) gives:

(4.3.9) Corollary. $M(x) = 2R(x) - R^2(x) - x$.

That is, $m(n) = 2r(n) - \sum_{k=1}^{n-1} r(k)\, r(n-k)$ if $n \geq 2$,

and $m(1) = 1$. //.

$r(n)$ may be evaluated recursively from the relationship $R(x) = x \exp\{\sum_{k=1}^{\infty} \frac{1}{k} R(x^k)\}$, due to Pólya (see [19, p.52]), and $m(n)$ is then evaluated using Corollary (4.3.9). The following table gives the values of $r(n)$ and $m(n)$ for $1 \leq n \leq 10$.

n	1	2	3	4	5	6	7	8	9	10
r(n)	1	1	2	4	9	20	48	115	286	719
m(n)	1	1	2	3	6	10	22	42	94	203

Das [8, Theorem 2] has given a partial solution to the problem of enumerating maximal connected topologies, stating that "the number of homeomorphism classes of maximal connected n-point topologies is equal to twice the number of n-point (*vertex*) trees minus the number of n-point (*vertex*) trees having a symmetry line." Although much of the approach is different, certain results of [8], some stated without proof, correspond to results presented here. For the purposes of studying and counting maximal connected topologies, the methods and terminology used here seem more concise and informative.

SECTION 4.4 The number of open sets of finite topologies

One of the more interesting problems in finite topologies asks: given n, for which values of r is there an n-point topology with r open sets? The study of the number of open sets, or *cardinality*, of topologies was introduced by Sharp [44], Stephen [47] and Stanley [46], and their investigations suggest the general question stated above. Although the problem seems exceedingly difficult to solve completely, this section provides some useful results, including existence and non-existence criteria, which lead to a solution for $n \leq 9$. Here, cardinalities of topologies are studied by considering their associated digraphs and it transpires that it is necessary to consider only those transitive digraphs which are partial orders. In conclusion, we put forward three conjectures concerning cardinality, which are based on the evidence available.

The cardinality of $(V, \mathcal{T})$ is denoted by $|\mathcal{T}|$. The set of values of r for which there is a topology of order n with cardinality r is the *spectrum* of n, written spec(n). The intention is to ascertain spec(n) for given n. Several results help to simplify the investigation of spec(n).

<u>(4.4.1) Lemma.</u> $\operatorname{spec}(n-1) \subset \operatorname{spec}(n)$.

<u>Proof</u>. Suppose $(V, \mathcal{T})$ is a topology of order n-1. For some $u \in V$, add a point $v \notin V$ to every open set of $\mathcal{T}$ containing u. This forms a topology $(V \cup v, \mathcal{T}')$ of order n with $|\mathcal{T}'| = |\mathcal{T}|$. So $\operatorname{spec}(n-1) \subseteq \operatorname{spec}(n)$. But the discrete topology of order n has cardinality 2^n whereas no topology of order n-1 can have cardinality more than 2^{n-1}. Thus $\operatorname{spec}(n-1) \subset \operatorname{spec}(n)$. //.

<u>(4.4.2) Lemma</u> [44]. If the separable topology $(V, \mathcal{T})$ is the disjoint union of connected topologies $(V_1, \mathcal{T}_1), \ldots, (V_k, \mathcal{T}_k)$ then $|\mathcal{T}| = |\mathcal{T}_1| \ldots |\mathcal{T}_k|$.

<u>Proof.</u> Since $V_1, \ldots, V_k$ are disjoint, the open sets of $\mathcal{T}$ are just those sets of the form $\bigcup_{i=1}^{k} A_i$ where $A_i \varepsilon \mathcal{T}_i$, and A_i can be chosen in $|\mathcal{T}_i|$ ways. //.

<u>(4.4.3) Corollary.</u> If $|\mathcal{T}| = p$, a prime, then $\mathcal{T}$ is connected, and hence $D(\mathcal{T})$ is weakly connected. //.

If D is the disjoint union of weakly connected digraphs $D_1, \ldots, D_k$ then $D_1, \ldots, D_k$ are the *weak components* of D. It follows from Corollary (4.2.4) and Lemma (4.3.1) that an alternative form of Lemma (4.4.2) is:

<u>(4.4.4) Corollary.</u> If the transitive digraph D has weak components $D_1, \ldots, D_k$ then $|\mathcal{T}(D)| = |\mathcal{T}(D_1)| \ldots |\mathcal{T}(D_k)|$. //.

For $r \geqslant 2$, m(r) is the smallest n for which there is a topology of order n with cardinality r. Lemma (4.4.1) implies that the general problem of determining spec(n) is equivalent to evaluating m(r) for all r. .Another consequence of Lemma (4.4.2) is:

<u>(4.4.5) Corollary.</u> For any factorisation of r into $r = r_1 \ldots r_k$,

$$m(r) \leqslant m(r_1) + \ldots + m(r_k).$$

<u>Proof.</u> Let $(V_i, \mathcal{T}_i)$ have order $m(r_i)$ and cardinality r_i in Lemma (4.4.2). Then $(V, \mathcal{T})$ has order $m(r_1) + \ldots + m(r_k)$ and cardinality r and the result follows. //.

Inequality is possible in Corollary (4.4.5), as will be shown later. It follows from Corollary (4.4.5) that:

<u>(4.4.6) Corollary.</u> For $r \notin$ spec(n-1), there is a separable topology of order n and cardinality r if and only if there is a factorisation of r into $r_1 \ldots r_k$ with $n = m(r_1) + \ldots + m(r_k)$.

<u>Proof.</u> Suppose $r \notin$ spec(n-1).

If $m(r_1) + \ldots + m(r_k) > n$ for all factorisations of r then there is no separable topology of order n with cardinality r. So there is a separable topology of order n with cardinality r only if $m(r_1) + \ldots + m(r_k) \leqslant n$ for some factorisation of r. But if $n > m(r_1) + \ldots + m(r_k)$ for any factorisation, then since, by Corollary (4.4.5), $m(r) \leqslant m(r_1) + \ldots + m(r_k)$, it follows that $m(r) < n$, contradicting the supposition that $r \notin \text{spec}(n-1)$. Thus for any factorisation of r, $m(r_1) + \ldots + m(r_k) \geqslant n$, and there is a separable topology of order n and cardinality r only if $n = m(r_1) + \ldots + m(r_k)$ for some factorisation. Clearly if $n = m(r_1) + \ldots + m(r_k)$ there is a separable topology of order n and cardinality r which is the disjoint union of topologies of order $m(r_i)$ and cardinality r_i. //.

<u>(4.4.7) Lemma.</u> If $m(r) = n$ then any topology $(V, \mathcal{T})$ of order n with cardinality r is T_0, and $D(\mathcal{T})$ is a partial order.

<u>Proof.</u> Assume that $m(r) = n$ and let $(V, \mathcal{T})$ of order n and cardinality r have mutually inseparable points u and υ. They can be identified as a single point in the open sets of $\mathcal{T}$ to form a topology of order n-1 with cardinality r, contradicting the assumption that $m(r) = n$. So u and υ cannot be mutually inseparable, hence $\mathcal{T}$ is T_0 and, by Proposition (4.1.2), $D(\mathcal{T})$ is a partial order. //.

If spec(1), ..., spec(n-1) are known, the cardinalities not belonging to spec(n-1) of the separable topologies of order n can be decided easily enough by using Corollary (4.4.6). Therefore in view of Lemma (4.4.1) and Corollary (4.4.6), to determine spec(n) it is sufficient to know

(1) spec(1), ..., spec(n-1)

(2) the cardinalities not belonging to spec(n-1) of the connected T_0 topologies of order n.

So spec(n) can be determined recursively by solving (2) for successive values of n. This is the approach taken here and henceforth all topologies are T_0 and all (transitive) digraphs are partial orders. The *cardinality* of a partial order means the cardinality of its associated topology.

Given an arbitrary partial order P it is difficult in general to evaluate its cardinality by inspection. The following basic theorem gives a method for obtaining the cardinality of a partial order as the sum of the cardinalities of two of its subgraphs. For any subset A of the vertex set V, P-A is the subgraph of P induced by V-A. G(υ) and L(υ) are as defined in section 1.1.

<u>(4.4.8) Theorem.</u> For any partial order P and any vertex υ of P,

(4.4.9) $|\mathcal{T}(P)| = |\mathcal{T}(P-\upsilon)| + |\mathcal{T}(P-\upsilon-G(\upsilon)-L(\upsilon))|$ and

(4.4.10) $|\mathcal{T}(P)| = |\mathcal{T}(P-\upsilon-G(\upsilon))| + |\mathcal{T}(P-\upsilon-L(\upsilon))|$.

<u>Proof.</u> The vertex set V of P can be partitioned into υ and the possibly empty subsets G(υ), L(υ) and V-υ-G(υ)-L(υ). Since P is a partial order, no vertex of V-υ-G(υ)-L(υ) is adjacent to any vertex of G(υ) and no vertex of L(υ) is adjacent to any vertex of V-υ-G(υ)-L(υ). Then

(1) The number of open sets of $\mathcal{T}(P)$ containing υ is $|\mathcal{T}(P-\upsilon-G(\upsilon))|$, since υ and G(υ) can be added to any open set of $\mathcal{T}(P-\upsilon-G(\upsilon))$ to form an open set of $\mathcal{T}(P)$ containing υ.

(2) The number of open sets of $\mathcal{T}(P)$ not containing υ is $|\mathcal{T}(P-\upsilon-L(\upsilon))|$, since any open set of $\mathcal{T}(P)$ containing a vertex of L(υ) will also contain υ.

From (1) and (2), $|\mathcal{T}(P)| = |\mathcal{T}(P-\upsilon-G(\upsilon))| + |\mathcal{T}(P-\upsilon-L(\upsilon))|$ and (4.4.10) is proved.

(3) The number of open sets of $\mathcal{T}(P-\upsilon)$ not containing any vertex of L(υ) is $|\mathcal{T}(P-\upsilon-L(\upsilon))|$.

(4) The number of open sets of $\mathcal{T}(P)$ containing υ but no vertex of

$L(\upsilon)$ is $|\mathcal{T}(P-\upsilon-G(\upsilon)-L(\upsilon))|$, since υ and $G(\upsilon)$ can be added to any open set of $\mathcal{T}(P-\upsilon-G(\upsilon)-L(\upsilon))$ to form an open set of $\mathcal{T}(P)$ containing υ.

(5) The number of open sets of $\mathcal{T}(P-\upsilon)$ containing a vertex of $L(\upsilon)$ equals the number of open sets of $\mathcal{T}(P)$ containing υ and a vertex of $L(\upsilon)$.

From (2) and (3), the number of open sets of $\mathcal{T}(P)$ not containing υ equals $|\mathcal{T}(P-\upsilon)|$ minus the number of open sets of $\mathcal{T}(P-\upsilon)$ containing a vertex of $L(\upsilon)$.

From (4), the number of open sets of $\mathcal{T}(P)$ containing υ equals $|\mathcal{T}(P-\upsilon-G(\upsilon)-L(\upsilon))|$ plus the number of open sets of $\mathcal{T}(P)$ containing υ and a vertex of $L(\upsilon)$. Adding these last two equations and using (5),

$$|\mathcal{T}(P)| = |\mathcal{T}(P-\upsilon)| + |\mathcal{T}(P-\upsilon-G(\upsilon)-L(\upsilon))| \quad \text{which is (4.4.9).} \qquad //.$$

The topology whose point set is empty is deemed to have cardinality one, the empty set, so that if $\upsilon \cup G(\upsilon) \cup L(\upsilon) = V$ then $|\mathcal{T}(P-\upsilon-G(\upsilon)-L(\upsilon))| = 1$. The generality and simplicity of Theorem (4.4.8) make it possible to evaluate $|\mathcal{T}(P)|$ very easily by choosing a suitable υ and applying (4.4.9) or (4.4.10), then repeating the procedure with the resultant subgraphs until the cardinalities can be determined by inspection.

The graph G is a *comparability graph* if it has a transitive orientation, which is a partial order.

<u>(4.4.11) Theorem.</u> All transitive orientations of a given comparability graph G have the same cardinality, that is if P_1 and P_2 are transitive orientations of G then $|\mathcal{T}(P_1)| = |\mathcal{T}(P_2)|$.

<u>Proof.</u> The proof is by induction on n, the order of G.
Theorem (4.4.11) is trivially true for $n = 1$ and suppose it to be true for all comparability graphs of order less than n. Let G be a comparability graph of order n with transitive orientations P_1 and P_2 and υ any

vertex of G. Now $G(P_1 - \upsilon)$ is isomorphic to $G(P_2 - \upsilon)$ and so the induction hypothesis implies that $|\mathcal{T}(P_1 - \upsilon)| = |\mathcal{T}(P_2 - \upsilon)|$. Further, since $G(\upsilon) \cup L(\upsilon)$ is identical in P_1 and P_2, $G(P_1 - \upsilon - G(\upsilon) - L(\upsilon))$ is isomorphic to $G(P_2 - \upsilon - G(\upsilon) - L(\upsilon))$ and by the hypothesis $|\mathcal{T}(P_1 - \upsilon - G(\upsilon) - L(\upsilon))| = |\mathcal{T}(P_2 - \upsilon - G(\upsilon) - L(\upsilon))|$. Then by (4.4.9), $|\mathcal{T}(P_1)| = |\mathcal{T}(P_2)|$. //.

$|G|$, the *cardinality* of G, means the cardinality of any transitive orientation of G.

<u>(4.4.12) Corollary.</u> For any comparability graph G and any vertex υ cf G,

$$|G| = |G-\upsilon| + |G-\upsilon-N(\upsilon)|$$

where $N(\upsilon)$, the *neighbourhood* of υ, is the set of vertices adjacent with υ.

<u>*Proof*</u>. *Corollary (4.4.12) follows from (4.4.9).* //.

The non-homeomorphic T_0 topologies with n points can be partially ordered by fineness. The first step in investigating the cardinalities of connected T_0 topologies of order n is to determine the cardinalities of all maximal connected topologies, and then the sequences of successively coarser connected T_0 topologies can be studied in detail. In view of Theorem (4.4.11) and Corollary (4.4.12) an equivalent procedure which is more convenient to use can be formulated in terms of comparability graphs. It follows from Theorem (4.1.3) that

(4.4.13) if $G_1 \subset G_2$ then $|G_1| > |G_2|$.

The alternative procedure is to look at the cardinalities of given connected comparability graphs and their successive transitively orientable minimal supergraphs, where G_2 is a (proper) *minimal supergraph* of G_1 if $G_1 \subset G_2$ and there is no graph G with $G_1 \subset G \subset G_2$. The starting points of these sequences are the trees of order n.

<u>(4.4.14) Theorem.</u> If T is a tree of order n then $f(n+1) \leqslant |T| \leqslant 2^{n-1} + 1$, where $f(n+1)$ is the Fibonacci number defined in section 2.2. Both bounds are attained.

<u>Proof.</u> The theorem is proved by induction on n.

It is true for $n = 1$ since the topology of order one has two open sets. Suppose the bounds are true for trees of order less than n, and let T be of order n. Let υ be a vertex of degree one, called an *endvertex* of T. Then $T-\upsilon$ is a tree and by assumption

(1) $\quad f(n) \leqslant |T-\upsilon| \leqslant 2^{n-2} + 1.$

Now $T-\upsilon-N(\upsilon)$ is an acyclic graph of order at most $n-2$ and hence by (4.4.13) and the induction hypothesis, $|T-\upsilon-N(\upsilon)| \geqslant f(n-1)$. Also, no topology of order $n-2$ has cardinality greater than the discrete topology, which has cardinality 2^{n-2}. Thus

(2) $\quad f(n-1) \leqslant |T-\upsilon-N(\upsilon)| \leqslant 2^{n-2}.$

But, by Corollary (4.4.12), $|T| = |T-\upsilon| + |T-\upsilon-N(\upsilon)|$, so adding (1) and (2), $f(n+1) \leqslant |T| \leqslant 2^{n-1} + 1$. Hence the bounds are generally true by induction.

The upper bound is attained by the tree with $n-1$ endvertices. If υ is the central vertex of degree $n-1$, $T-\upsilon$ is totally disconnected and so $\mathcal{T}(T-\upsilon)$ is the discrete topology of order $n-1$ with cardinality 2^{n-1}. $|T-\upsilon-N(\upsilon)| = 1$ since $\upsilon \cup N(\upsilon)$ is the vertex set of T. Thus $|T| = 2^{n-1} + 1$.

The lower bound is attained by the tree with two endvertices, so that all other vertices have degree 2. If $T(n)$ is the tree of order n with two endvertices and υ is one of the endvertices, $T-\upsilon = T(n-1)$ and $T-\upsilon-N(\upsilon) = T(n-2)$. Therefore $|T(n)| = |T(n-1)| + |T(n-2)|$ and so $|T(n)| = f(n+1)$. //.

So a connected topology of order n has cardinality at most $2^{n-1} + 1$, as shown in [44, Theorem 2], and a maximal connected topology of order n has cardinality between $f(n+1)$ and $2^{n-1} + 1$. The cardinalities of the maximal connected topologies of orders up to 9 are listed together with their tree diagrams in Appendix 3. Several criteria for existence and non-existence of cardinalities of order n can now be established, after which it will be possible to derive spec(n) for orders up to 9.

(4.4.15) Theorem. If T is a tree of order n, then

(1) $|T| = 2^{n-1} + 1$ and $T = T_1(n)$, the tree with one vertex of degree n-1 and n-1 endvertices,

or (2) $|T| = 3.2^{n-3} + 2$ and $T = T_2(n)$, the tree with one vertex of degree n-2, one vertex of degree 2 and n-2 endvertices,

or (3) $|T| \leq 5.2^{n-4} + 4$.

Proof. Suppose T has vertex υ with greatest degree d. Then $d \leq n-1$.

If $d = n-1$, $T = T_1(n)$ and it has been shown in the proof of Theorem (4.4.14) that $|T_1(n)| = 2^{n-1} + 1$.

If $d = n-2$, $T = T_2(n)$, and, by Corollary (4.4.12), $|T_2(n)| = |T_2(n)-\upsilon| + |T_2(n)-\upsilon-N(\upsilon)| = 3.2^{n-3} + 2$.

If $d = 2$, $T = T(n)$ as defined in the proof of Theorem (4.4.14). For $n \leq 3$, $T(n) = T_1(n) = T_2(n)$ and for $n \geq 4$, $|T(n)| = f(n+1) < 5.2^{n-4} + 4$.

Otherwise $3 \leq d \leq n-3$.

Now $T-\upsilon$ has d components and n-d-1 edges. Let the components of $T-\upsilon$ be $T_1, \ldots, T_d$ of orders $r_1, \ldots, r_d$. Then $|T-\upsilon| = |T_1| \ldots |T_d|$ and, by Theorem (4.4.14), $|T-\upsilon|$ is greatest when $|T_i| = 2^{r_i-1} + 1$ for all $i = 1, \ldots, d$. Further, since $2(2^{r_1+r_2-2} + 1) \geq (2^{r_1-1}+1)(2^{r_2-1} + 1)$ with inequality unless r_1 or r_2 is equal to 1, $|T-\upsilon|$ is a maximum when $r_1 = n-d$ and $r_2 = \ldots = r_d = 1$.

Hence $|T-\upsilon| \leq 2^{d-1}(2^{n-d-1} + 1)$. $T-\upsilon-N(\upsilon)$ is an acyclic graph of order n-d-1 and so $|T-\upsilon-N(\upsilon)| \leq 2^{n-d-1}$. Thus $|T| = |T-\upsilon| + |T-\upsilon-N(\upsilon)| \leq 2^{d-1}(2^{n-d-1} + 1) + 2^{n-d-1} = 2^{n-2} + 2^{d-1} + 2^{n-d-1}$. But since $3 \leq d \leq n-3$, $|T| \leq 2^{n-2} + 2^{3-1} + 2^{n-3-1} = 5.2^{n-4} + 4$ and the proof is complete. //.

Equality holds in part (3) of Theorem (4.4.15). For let $T_3(n)$ have one

vertex υ of degree n-3, one vertex of degree 3 and n-2 endvertices.
Then $|T_3(n)| = |T_3(n)-\upsilon| + |T_3(n)-\upsilon-N(\upsilon)| = 5.2^{n-4} + 4.$

The next corollary is an important non-existence criterion for determining whether a given $r \in spec(n)$.

(4.4.16) Corollary   If G is a connected comparability graph of order n, then

(1) $|G| = 2^{n-1} + 1$ and $G = T_1(n)$,

or (2) $|G| = 3.2^{n-3} + 2$ and $G = T_2(n)$,

or (3) $|G| = 3.2^{n-3} + 1$ and $G = C_1(n)$, the comparability graph formed by adding an edge to $T_1(n)$ between two endvertices,

or (4) $|G| \leqslant 5.2^{n-4} + 4.$

Proof.   It follows from Theorem (4.4.15) that if G is a connected comparability graph of order n with cardinality greater than $5.2^{n-4} + 4$ then $T_1(n) \subseteq G$ or $T_2(n) \subseteq G$.
Consider first the (proper) supergraphs of $T_1(n)$. The only minimal supergraph of $T_1(n)$ is the comparability graph $C_1(n)$ and $|C_1(n)| = 3.2^{n-3} + 1.$
The minimal supergraphs of $C_1(n)$ are $C_2(n)$, formed by adding an edge between two endvertices, and $C_3(n)$, formed by adding an edge between an endvertex and a vertex of degree 2. Both $C_2(n)$ and $C_3(n)$ are comparability graphs but $|C_2(n)| = 3^2.2^{n-5} + 1$ and $|C_3(n)| = 5.2^{n-4} + 1$, both less than $5.2^{n-4} + 4$.
The minimal supergraphs of $T_2(n)$ are $C_1(n)$, formed by adding an edge between the vertex of degree n-2 and the endvertex adjacent with the vertex of degree 2; $C_4(n)$, formed by adding an edge between an endvertex adjacent with the vertex of degree n-2 and the endvertex adjacent with the vertex of degree 2; $C_5(n)$, formed by adding an edge between the vertex of degree 2 and an endvertex adjacent with the vertex of degree n-2; and $C_6(n)$, formed by adding an edge between two of the endvertices adjacent with the vertex of degree n-2. $C_4(n)$, $C_5(n)$ and

$C_6(n)$ are comparability graphs but $|C_4(n)| = 5.2^{n-4} + 2$, $|C_5(n)| = 5.2^{n-4} + 2$ and $|C_6(n)| = 3^2.2^{n-5} + 2$, all less than $5.2^{n-4} + 4$. //.

Theorem (4.4.15) and Corollary (4.4.16) can be extended as necessary but this becomes a tedious and lengthy process. There seems to be no simple test of non-existence.

(4.4.17) Theorem. For $n \geqslant 2$,

(1) $r \in \text{spec}(n)$,

and (2) some partial order P of order n with cardinality r is weakly connected,

if and only if

(3) there is a weakly connected partial order P_1 of order n-1 with cardinality greater than $r/2$,

(4) P_1 has an induced subgraph P_2 with $|T(P_2)| = r-|T(P_1)|$, so that $r-|T(P_1)| \in \text{spec}(n-2)$,

and (5) $V(P_1) - V(P_2)$ can be partitioned into subsets A and B with no vertex of A adjacent from any vertex of P_2; no vertex of B adjacent to any vertex of P_2; and every vertex of A adjacent to every vertex of B.

Proof. First suppose that (1) and (2) are true. As P is weakly connected then P-υ is weakly connected for some $\upsilon \in V(P)$. Put $P_1 = P-\upsilon, P_2 = P-\upsilon-G(\upsilon)-L(\upsilon)$, $A = G(\upsilon)$ and $B = L(\upsilon)$. Then since $n \geqslant 2$, $P_2 \subset P_1$ and so $|T(P_1)| > |T(P_2)|$. By (4.4.9), $r = |T(P)| = |T(P_1)| + |T(P_2)|$. Hence $|T(P_1)| > r/2$, $|T(P_2)| = r-|T(P_1)|$ and (3) and (4) are proved. (5) is true because P is a partial order.

Conversely, suppose (3), (4) and (5) are true. Add a vertex $\upsilon \notin V(P_1)$ to P_1 to form P, where υ is adjacent from every vertex of A and is adjacent to every vertex of B. Then $G(\upsilon) = A$, $L(\upsilon) = B$. (5) ensures that P is a partial order. Since $V(P_1) - V(P_2) \neq \phi$, $A \cup B = G(\upsilon) \cup L(\upsilon)$ is not empty and P is weakly connected. Further, $P_1 =$

$P-v$ and $P_2 = P-v-G(v)-L(v)$ so $|\mathcal{T}(P)| = |\mathcal{T}(P-v)| + |\mathcal{T}(P-v-G(v)-L(v))| = |\mathcal{T}(P_1)| + |\mathcal{T}(P_2)| = r$. Therefore (1) and (2) are true. //.

Theorem (4.4.17) is used as a constructive test of whether there is a partial order of order n with given cardinality r when it is known that any such partial order must be weakly connected. As a test of non-existence it is not always practicable, perhaps requiring a search of all weakly connected partial orders of order n-1 with cardinality between r/2 and r-1.

<u>(4.4.18) Lemma.</u> If $r \in \text{spec}(n-1)$ then $r + 1 \in \text{spec}(n)$.

<u>Proof.</u> Let G_1 be a comparability graph of order n-1 with cardinality r. Form G of order n by adding $v \notin V(G_1)$ to G_1 with $N(v) = V(G_1)$. G is a comparability graph of order n since G_1 has a transitive orientation and v can be a source. As $V(G-v-N(v)) = \phi$, then $|G| = |G-v| + |G-v-N(v)| = |G_1| + 1 = r + 1$, so $r + 1 \in \text{spec}(n)$. //.

<u>(4.4.19) Lemma.</u> If a comparability graph of order n-1 with cardinality r has an endvertex, then $r+3 \in \text{spec}(n)$.

<u>Proof.</u> Let the comparability graph G_1 of order n-1 with cardinality r have endvertex u adjacent with ω. Form G of order n by adding $v \notin V(G_1)$ to G_1 with $N(v) = V(G_1)-u-\omega$. Let P_1 be a transitive orientation of G_1. Then G has a transitive orientation, with the edges of G_1 oriented as in P_1 and with v as a transmitter if $\omega R u$ in P_1 or as a receiver if $u R \omega$. $G-v-N(v)$ is the tree of order 2 and it follows from Theorem (4.4.14) that $|G-v-N(v)| = 3$.
Thus G is a comparability graph with $|G| = |G-v| + |G-v-N(v)| = |G_1| + 3 = r + 3$, so $r + 3 \in \text{spec}(n)$. //.

<u>(4.4.20) Lemma.</u> If a comparability graph of order n-1 with

cardinality r has two endvertices adjacent with the same vertex, then $r + 5 \in$ spec(n).

Proof. Let the comparability graph G_1 of order n-1 with cardinality r have endvertices u and ω adjacent with z. Form G of order n by adding $\upsilon \notin V(G_1)$ to G_1 with $N(\upsilon) = V(G_1)-u-\omega-z$. Let P_1 be a transitive orientation of G_1. G has a transitive orientation, with the edges of G_1 oriented as in P_1 and with υ as a transmitter if z is a transmitter in P_1 or as a receiver if z is a receiver. $G-\upsilon-N(\upsilon)$ is the tree of order 3 and it follows from Theorem (4.4.14) that $|G-\upsilon-N(\upsilon)| = 5$. G is a comparability graph with $|G| = |G-\upsilon| + |G-\upsilon-N(\upsilon)| = |G_1| + 5 = r + 5$. So $r + 5 \in$ spec(n). //.

It is now possible to ascertain spec(n) for $1 \leqslant n \leqslant 8$ using the results already derived. There is some overlap between these criteria and it is often the case that any one of several can be invoked in order to decide whether a given $r \in$ spec(n). The notation $\{m \rightarrow n\}$ means the set of integers in the (closed) interval $[m, n]$.

n = 1. spec(1) = {2} since the only topology with one point is the (in) discrete topology.

n = 2. 2 $\in$ spec(2) by (4.4.1).

4 $\in$ spec(2) by (4.4.6).

Otherwise $\mathcal{T}$ is T_0 and connected, and so $|\mathcal{T}| \leqslant 3$ by (4.4.14).

3 $\in$ spec(2) by (4.4.18).

Hence spec(2) = $\{2 \rightarrow 4\}$.

n = 3. $\{2 \rightarrow 4\} \subset$ spec(3) by (4.4.1).

$\{6, 8\} \subset$ spec(3) by (4.4.6).

Otherwise $\mathcal{T}$ is T_0 and connected, and so $|\mathcal{T}| \leqslant 5$ by (4.4.14).

$5 \in \mathrm{spec}(3)$ by (4.4.18).

Hence $\mathrm{spec}(3) = \{2 \to 6, 8\}$.

<u>n = 4.</u> $\{2 \to 6, 8\} \subset \mathrm{spec}(4)$ by (4.4.1).

$\{9, 10, 12, 16\} \subset \mathrm{spec}(4)$ by (4.4.6).

Otherwise $\mathcal{T}$ is T_0 and connected, and so $|\mathcal{T}| \leqslant 9$ by (4.4.14).

$7 \in \mathrm{spec}(4)$ by (4.4.18).

Hence $\mathrm{spec}(4) = \{2 \to 10, 12, 16\}$.

<u>n = 5.</u> $\{2 \to 10, 12, 16\} \subset \mathrm{spec}(5)$ by (4.4.1).

$\{14, 15, 18, 20, 24, 32\} \subset \mathrm{spec}(5)$ by (4.4.6).

Otherwise $\mathcal{T}$ is T_0 and connected, and so $|\mathcal{T}| \leqslant 17$ by (4.4.14).

$\{11, 13, 17\} \subset \mathrm{spec}(5)$ by (4.4.18).

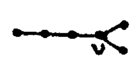
Hence $\mathrm{spec}(5) = \{2 \to 18, 20, 24, 32\}$.

<u>n = 6.</u> $\{2 \to 18, 20, 24, 32\} \subset \mathrm{spec}(6)$ by (4.4.1).

$\{21, 22, 25 \to 28, 30, 34, 36, 40, 48, 64\} \subset \mathrm{spec}(6)$ by (4.4.6).

Otherwise $\mathcal{T}$ is T_0 and connected, and so from (4.4.16),
$|\mathcal{T}| = 33$, $|\mathcal{T}| = 26$, $|\mathcal{T}| = 25$ or $|\mathcal{T}| \leqslant 24$.

$19 \in \mathrm{spec}(6)$ by (4.4.18).

$23 \in \mathrm{spec}(6)$ by (4.4.19), with $G_1 =$  and $G =$ .

Hence $\mathrm{spec}(6) = \{2 \to 28, 30, 32 \to 34, 36, 40, 48, 64\}$.

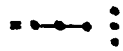
This answers in the negative a question of Sharp [44] who asked: given any prime number $p \leqslant 2^{n-1} + 1$, is there a topology of order n with cardinality p? For $n = 6$, $p = 29$ and $p = 31$ are counterexamples.

<u>n = 7.</u> $\{2 \to 28, 30, 32 \to 34, 36, 40, 48, 64\} \subset \mathrm{spec}(7)$ by (4.4.1).

$\{35, 38, 39, 42, 44 \to 46, 50 \to 52, 54, 56, 60, 66, 68, 72, 80, 96, 128\} \subset \mathrm{spec}(7)$ by (4.4.6).

Otherwise $\mathcal{T}$ is T_0 and connected, and so from (4.4.16), $|\mathcal{T}| = 65$, $|\mathcal{T}| = 50$, $|\mathcal{T}| = 49$ or $|\mathcal{T}| \leqslant 44$.

$\{29, 31, 37, 41\} \subset \mathrm{spec}(7)$ by (4.4.18).

$43 \in \mathrm{spec}(7)$ by (4.4.19), with G_1 =  and G = 

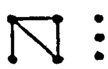
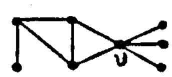
Hence $\mathrm{spec}(7) = \{2 \to 46, 48 \to 52, 54, 56, 60, 64 \to 66, 68, 72, 80, 96, 128\}$.

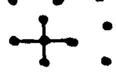
<u>n = 8.</u> $\{2 \to 46, 48 \to 52, 54, 56, 60, 64 \to 66, 68, 72, 80, 96, 128\} \subset \mathrm{spec}(8)$ by (4.4.1).

$\{55, 57, 58, 62, 63, 69, 70, 74 \to 76, 78, 81, 82, 84 \to 86, 88, 90, 92, 98 \to 100, 102, 104, 108, 112, 120, 130, 132, 136, 144, 160, 192, 256\} \subset \mathrm{spec}(8)$ by (4.4.6).

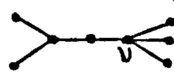
Otherwise $\mathcal{T}$ is T_0 and connected, and so from (4.4.16), $|\mathcal{T}| = 129$, $|\mathcal{T}| = 98$, $|\mathcal{T}| = 97$ or $|\mathcal{T}| \leqslant 84$.

$\{47, 53, 61, 67, 73\} \subset \mathrm{spec}(8)$ by (4.4.18).

$59 \in \mathrm{spec}(8)$ by (4.4.19), with G_1 =  and G = 

$71 \in \mathrm{spec}(8)$ by (4.4.19), with G_1 =  and G = 

$83 \in \mathrm{spec}(8)$ by (4.4.19), with G_1 =  and G = 

$77 \in \mathrm{spec}(8)$ by (4.4.20), with G_1 =  and G = 

This leaves $\{79\}$ as the only undecided case.

<u>(4.4.21) Lemma.</u> There is no topology of order 8 with cardinality 79.

<u>Proof.</u> The proof involves applying Theorem (4.4.17). It has been shown that any topology of order 8 with cardinality 79 is T_0 and connected. Then by Theorem (4.4.17), in order that $79 \in \mathrm{spec}(8)$:

(1) there is a weakly connected partial order P_1 of order 7 with cardinality $r \geqslant 40$,

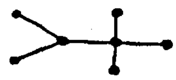
and (2) P_1 has an induced subgraph P_2 with cardinality 79-r, where

$79-r \in \text{spec}(6)$.

The only possible values of r are 65, 49 and 43.

<u>Case 1. $r = 65$.</u> Then $G(P_1) = T_1(7)$ by Corollary (4.4.16). But $T_1(7)$ has no induced subgraph with cardinality 14, so $r = 65$ is impossible.

<u>Case 2. $r = 49$.</u> Then $G(P_1) = C_1(7)$ by Corollary (4.4.16). $C_1(7)$ has no induced subgraph with cardinality 30, so $r = 49$ is impossible.

<u>Case 3. $r = 43$.</u> From the proof of Corollary (4.4.16) it follows that if $|T(P_1)| = 43$, $G(P_1)$ cannot be a supergraph of $C_1(7)$ or $T_2(7)$. Further, since $G(P_1)$ must be connected reference to Appendix 3 shows that $G(P_1) =$  or $G(P_1) \supset$ 
By inspection,  has no supergraph of cardinality greater than 42. Thus  is the only connected graph of order 7 with cardinality 43 and by inspection it has no induced subgraph $G(P_2)$ with cardinality 36. So $r = 43$ is not possible.

Altogether, $79 \notin \text{spec}(8)$. //.

Hence $\text{spec}(8) = \{2 \to 78, 80 \to 86, 88, 90, 92, 96 \to 100, 102, 104, 108, 112, 120, 128 \to 130, 132, 136, 144, 160, 192, 256\}$.

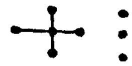
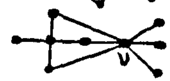
<u>$n = 9$.</u> $\{2 \to 78, 80 \to 86, 88, 90, 92, 96 \to 100, 102, 104, 108, 112, 120, 128 \to 130, 132, 136, 144, 160, 192, 256\} \subset \text{spec}(9)$ by (4.4.1).

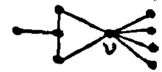
$\{87, 91, 93 \to 95, 105, 106, 110, 111, 114 \to 119, 122 \to 126, 134, 135, 138, 140, 142, 146, 148, 150, 152 \to 154, 156, 162, 164 \to 166, 168, 170, 172, 176, 180, 184, 194 \to 196, 198, 200, 204, 208, 216, 224, 240, 258, 260, 264, 272, 288, 320, 384, 512\} \subset \text{spec}(9)$ by (4.4.6).

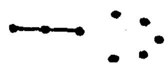
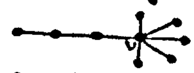
Otherwise T is T_0 and connected, and so from (4.4.16), $|T| = 257$, $|T| = 194$, $|T| = 193$ or $|T| \leq 164$.

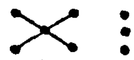
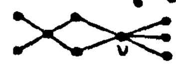
{79, 89, 101, 103, 109, 113, 121, 131, 133, 137, 145, 161} ⊂ spec(9) by (4.4.18).

107 ε spec(9) by (4.4.19), with G_1 =  and G = 

139 ε spec(9) by (4.4.19), with G_1 =  and G = 

147 ε spec(9) by (4.4.19), with G_1 =  and G = 

163 ε spec(9) by (4.4.19), with G_1 =  and G = 

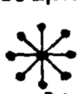
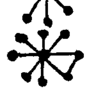
141 ε spec(9) by (4.4.20), with G_1 =  and G = 

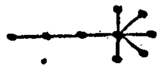
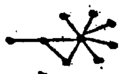
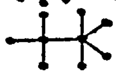
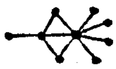
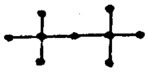
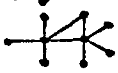
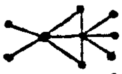
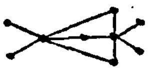
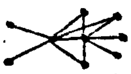
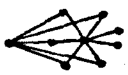
149 ε spec(9) by (4.4.20), with G_1 =  and G = 

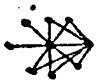
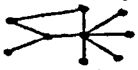
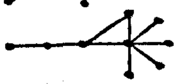
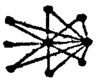
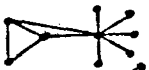
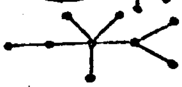
It remains to determine the cases 127, 143, 151, 155, 157, 158 and 159.

<u>(4.4.22). Lemma.</u> For r ε {127, 143, 151, 155, 157, 158, 159} there is no topology of order 9 with cardinality r.

<u>Proof.</u> The proof is an extension of Corollary (4.4.16). It has been shown that any topology of order 9 with cardinality r ε {127, 143, 151, 155, 157, 158, 159} is T_0 and connected. Any comparability graph of order 9 with cardinality r ε {127, 143, 151, 155, 157, 158, 159} is therefore a supergraph of a tree of order 9 with cardinality not less than 127. There are 11 such trees, as shown in Appendix 3. By considering the sequences of minimal supergraphs of these 11 trees it is shown that there is no comparability graph of order 9 with cardinality r ε {127, 143, 151, 155, 157, 158, 159}. Since every non-empty comparability graph is a minimal supergraph of another comparability graph, those minimal supergraphs which are not comparability graphs can be ignored. The comparability graphs are numbered in order of decreasing cardinality and those with cardinality at least 127 have their minimal supergraphs listed alongside.

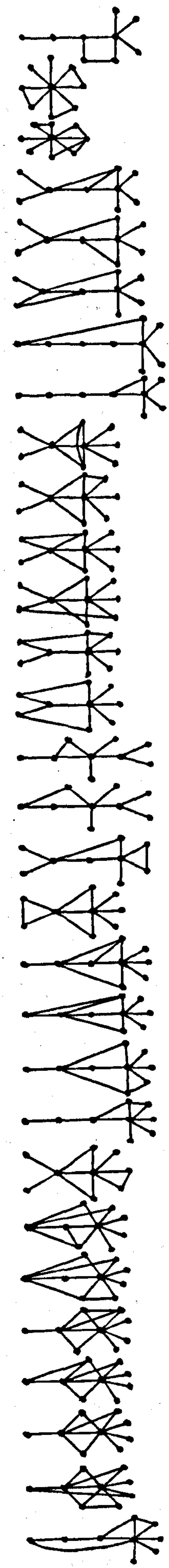
Number	Graph	Cardinality	Minimal supergraphs
1		257	3
2		194	3, 6, 7, 14
3		193	8, 18

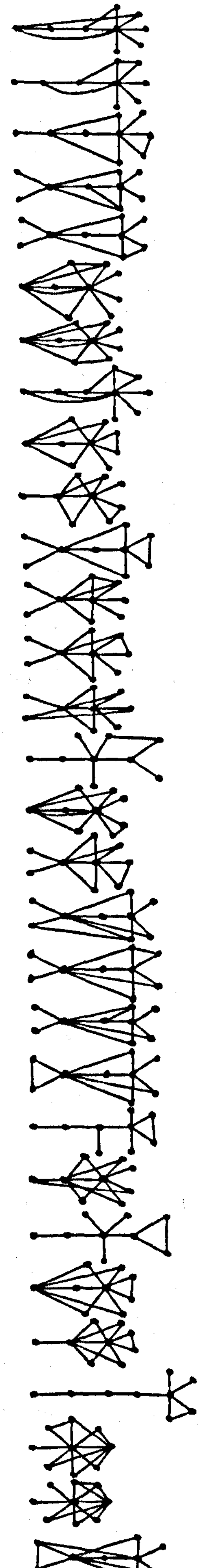
Number	Graph	Cardinality	Minimal supergraphs
4		164	7, 12, 36, 40, 51
5		163	6, 7, 13, 40, 41, 54
6		162	8, 15, 44, 45, 55
7		162	8, 16, 44, 45, 46, 56
8		161	19, 48, 49, 58
9		152	12, 20, 59, 61, 69
10		149	12, 13, 21, 65, 66, 70, 92
11		148	14, 36, 37, 52, 53, 98
12		148	16, 22, 71, 72, 73, 78, 99
13		147	15, 16, 23, 79, 80, 81, 104
14		146	18, 46, 55, 56, 57, 106
15		146	19, 24, 84, 85, 107
16		146	19, 25, 86, 87, 88, 89, 108
17		145	20, 21, 93, 105
18		145	49, 58, 112
19		145	27, 94, 95, 113
20		144	22, 100, 101, 102, 109, 110
21		141	22, 23, 26, 114, 115, 116, 127
22		140	25, 28, 119, 120, 121, 122, 128, 133
23		139	24, 25, 29, 129, 130, 131, 143
24		138	27, 31, 134, 135, 149
25		138	27, 32, 136, 137, 138, 139, 150
26		137	28, 29, 144, 145, 151
27		137	34, 146, 147, 156
28		136	32, 152, 153, 154, 157
29		135	31, 32, 35, 158, 159, 160, 161
30		134	36, 37, 41, 62, 63, 64, 66, 67, 111, 162
31		134	34, 38, 165

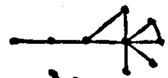
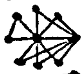
Number	Graph	Cardinality	Minimal supergraphs
32		134	34, 39, 163, 166
33		133	36, 41, 66, 68, 70, 117, 118, 167
34		133	42, 168, 169, 172
35		133	38, 39, 170, 171,
36		132	45, 46, 71, 73, 74, 75, 80, 123 124, 173
37		132	46, 71, 76, 82, 123, 174
38		132	42, 43, 175, 176
39		132	42, 177, 178, 179
40		131	44, 78, 80, 180
41		131	45, 46, 78, 79, 81, 82, 83 132, 181
42		131	47, 182, 183
43		131	47, 184
44		130	48, 84, 87, 185
45		130	49, 84, 85, 87, 89, 90, 140 141, 186
46		130	49, 86, 87, 88, 90, 91, 140 142, 187
47		130	188
48		129	96, 190
49		129	94, 95, 97, 148, 191
50		128	51, 52, 59, 60, 77, 103, 125 126, 155, 164, 189

Number	Graph	Cardinality
51		124
52		124
53		124
54		123
55		122
56		122
57		122
58		121
59		120
60		120
61		118
62		118
63		118
64		118
65		117
66		117
67		117
68		117
69		116
70		116
71		116
72		116
73		116
74		116
75		116
76		116
77		116
78		115
79		115
80		115
81		115
82		115
83		115
84		114
85		114
86		114
87		114
88		114
89		114
90		114
91		114
92		113
93		113
94		113
95		113
96		113
97		113
98		112
99		112
100		112
101		112
102		112
103		112
104		111
105		111
106		110
107		110
108		110
109		110
110		110

Number	Graph	Cardinality	Number	Graph	Cardinality
111		110	141		106
112		109	142		106
113		109	143		105
114		109	144		105
115		109	145		105
116		109	146		105
117		109	147		105
118		109	148		105
119		108	149		104
120		108	150		104
121		108	151		104
122		108	152		104
123		108	153		104
124		108	154		104
125		108	155		104
126		108	156		103
127		107	157		103
128		107	158		103
129		107	159		103
130		107	160		103
131		107	161		102
132		107	162		102
133		106	163		102
134		106	164		102
135		106	165		101
136		106	166		101
137		106	167		101
138		106	168		101
139		106	169		101
140		106	170		101





Number	Graph	Cardinality	Number	Graph	Cardinality
171		101	181		99
172		100	182		99
173		100	183		99
174		100	184		99
175		100	185		98
176		100	186		98
177		100	187		98
178		100	188		98
179		100	189		98
180		99	190		97
			191		97

So there are no comparability graphs of order 9 with cardinality 127, 143, 151, 155, 157, 158 or 159. //.

Hence spec(9) = {2 → 126, 128 → 142, 144 → 150, 152 → 154, 156, 160 → 166, 168, 170, 172, 176, 180, 184, 192 → 196, 198, 200, 204, 208, 216, 224, 240, 256 → 258, 260, 264, 272, 288, 320, 384, 512}.

Parchmann [36] has independently derived much of the above information, including Lemma (4.4.18). For n ≤ 10, he has programmed a computer to list the cardinalities of all T_0 topologies of order n with generating topology on at most three points. For n ≤ 9 the list of such topologies of order n is identical to spec(n), so we have established that there are no cardinalities other than those listed in [36].

Examination of spec(1), ..., spec(9) reveals some interesting patterns in the presence or absence of particular cardinalities. To conclude the section, three conjectures about cardinality are stated. Although these conjectures appear to be difficult to prove in general, there is some evidence in favour of them, being true for orders up to and including nine.

It was remarked earlier that inequality is possible in Corollary (4.4.5). Examples are

$$m(33) = 6 < 2 + 5 = m(3) + m(11)$$

$$m(49) = 7 < 4 + 4 = m(7) + m(7)$$

$$m(65) = 7 < 3 + 5 = m(5) + m(13)$$

$$m(77) = 8 < 4 + 5 = m(7) + m(11)$$

However, no value of $r \geq 2$ has yet been found for which $m(2r) < m(2) + m(r)$. Since $m(2) = 1$, $m(r) \leq m(2r) \leq m(2) + m(r)$ implies that either $m(2r) \leq m(r)$ or $m(2r) = 1 + m(r)$. Reference to spec(1), ..., spec(9) shows no instance where $m(2r) \leq m(r)$ and for $r = 2, \ldots, 126$, $m(2r) = 1 + m(r)$. It seems reasonable to make the following conjecture:

<u>(4.4.23) Conjecture.</u> For $r \geq 2$, $m(2r) = 1 + m(r)$. Equivalently, for $r \geq 2$, $2r \in \text{spec}(n)$ if and only if $r \in \text{spec}(n-1)$.

Since $m(2r) \leq 1 + m(r)$, (4.4.23) can be reduced to:

<u>(4.4.24) Conjecture.</u> If G is a comparability graph of order n with $|G| \geq 4$ and even, then there is a comparability graph G_1 of order $n - 1$ with $|G| = 2|G_1|$.

A restricted case of (4.4.24) is:

<u>(4.4.25) Proposition.</u> If a comparability graph G of order n has endvertices u and υ distance three apart, then there is a comparability graph G_1 of order $n - 1$ with $|G| = 2|G_1|$.

<u>Proof.</u> Let u and υ be adjacent with x and y respectively in G. Then x and y are adjacent in G. Form G_2 from G by deleting the edge between u and x and inserting an edge between x and υ. It is easily shown that G_2 is also a comparability graph. Now $G - x \cong G_2 - x$. Moreover, letting $N_2(x)$ be the neighbourhood of x in G_2, $G - x - N(x)$ is isomorphic to $G_2 - x - N_2(x)$ since the isolated vertex υ in $G - x - N(x)$ is exchanged for the isolated vertex u in $G_2 - x - N_2(x)$. Since $|G| = |G-x| + |G-x-N(x)|$ and $|G_2| = |G_2-x| + |G_2-x-N_2(x)|$ from Corollary (4.4.12), then $|G| = |G_2|$. Define G_1 as G_2-u. As u is an isolated vertex in G_2, so G_1 is a comparability graph of order $n - 1$

with $|G_2| = 2|G_1|$. Hence $|G| = 2|G_1|$. //.

Suppose P is a transitive orientation of G with P_1 as the corresponding transitive orientation of G_1. In topological terms, if υRx in P_1 then to transform $(V(P), \mathcal{T}(P))$ to $(V(P_1), \mathcal{T}(P_1))$ delete u from $\mathcal{T}(P)$ and add υ to any open set of $\mathcal{T}(P\ )$ containing x. Alternatively, if $xR\upsilon$ in P_1 delete u from $\mathcal{T}(P\ )$ and add x to any open set of $\mathcal{T}(P\ )$ containing υ. Some examples of Proposition (4.4.25) are, using the notation introduced in Theorem (4.4.15) and Corollary (4.4.16), $T_2(n) = 2|C_1(n-1)|$, $|C_6(n)| = 2|C_2(n-1)|$, $|C_5(n)| = 2|C_3(n-1)|$ and $|T_3(n)| = 2|C_5(n-1)|$.

The following result provides a recursive test of whether $|G|$ is even when the comparability graph G has an endvertex. Define $D_r(\upsilon)$ to be the set of vertices distance r from the vertex υ in G, so that $D_0(\upsilon) = \{\upsilon\}$ and $D_1(\upsilon) = N(\upsilon)$.

<u>(4.4.26) Proposition.</u> If the comparability graph G has an endvertex υ, then $|G|$ has the same parity as $|G - D_0(\upsilon) - D_1(\upsilon) - D_2(\upsilon)|$.

<u>Proof.</u> Let υ be adjacent with u. Then $D_1(\upsilon) = u$ and $D_2(\upsilon) = N(u) - \upsilon$. Now υ is an isolated vertex in G - u, so $|G-u| = 2|G-u-\upsilon|$ and therefore $|G-u|$ is even. But from Corollary (4.4.12), $|G| = |G-u| + |G-u-N(u)|$. Hence $|G|$ has the same parity as $|G-u-N(u)|$, and G-u-N(u) is $G-D_0(\upsilon) - D_1(\upsilon) - D_2(\upsilon)$. //.

Parchmann [36] has postulated an equivalent version of (4.4.24) and a less general form of the following conjecture which is true for $n \leqslant 9$.

<u>(4.4.27) Conjecture.</u> If r_1 and r_2 are members of spec(n) where $r_1 < r_2$ and there is no $r \in$ spec(n) for any $r_1 < r < r_2$, then $r_1 = k2^l$ and $r_2 = (k+1)2^l$ for some $k \geqslant 1$ and $l \geqslant 0$. That is, consecutive members of spec(n) are of the form $k2^l$ and $(k+1)2^l$ for some $k \geqslant 1$ and $l \geqslant 0$.

Now let $c(n)$ be the greatest integer such that $\{2 \to c(n)\} \subseteq spec(n)$. $c(n) + 1$ is the smallest integer greater than one which is not a member of $spec(n)$. A table of values of $c(n)$ for $n \leqslant 9$ is:

n	1	2	3	4	5	6	7	8	9
c(n)	2	4	6	10	18	28	46	78	126

<u>(4.4.28) Conjecture.</u> For $n \geqslant 3$, $c(n) \geqslant c(n-1) + c(n-2)$.

The final result provides a lower bound for $c(n)$.

<u>(4.4.29) Proposition.</u> For $n \geqslant 3$, $c(n) \geqslant 2[c(n-2) + 1]$.

<u>Proof.</u> By Lemma (4.4.1), $\{2 \to c(n-2)\} \subseteq spec(n-2)$ implies that $\{2 \to c(n-2)\} \subseteq spec(n)$.

By Corollary (4.4.6), $\{2 \to c(n-2)\} \subseteq spec(n-2)$ implies that all even numbers from $c(n-2) + 1$ to $2c(n-2)$ inclusive are members of $spec(n-1)$, which in turn implies that $\{c(n-2) + 1 \to 2c(n-2) + 1\} \subseteq spec(n)$ by Lemma (4.4.1) and Lemma (4.4.18).

Further, $c(n-2) \in spec(n-2)$ implies that $c(n-2) + 1 \in spec(n-1)$ from Lemma (4.4.18), so that by Corollary (4.4.6), $2[c(n-2) + 1] \in spec(n)$.

Altogether, $\{2 \to 2[c(n-2) + 1]\} \subseteq spec(n)$, so $c(n) \geqslant 2[c(n-2) + 1]$. //.

Proposition (4.4.29) can be stated explicitly as

$$c(n) \geqslant 2\sqrt{2}.2^{n/2} - 2 \text{ if } n \text{ is odd,}$$

and

$$c(n) \geqslant 3.2^{n/2} - 2 \text{ if } n \text{ is even,}$$

which is the same as [36, Theorem 5].

This lower bound for $c(n)$ is not sharp as comparison of the following table with that for $c(n)$ shows.

n	3	4	5	6	7	8	9
Lower bound for c(n)	6	10	14	22	30	46	62

APPENDIX 1

Table 1

The number of score sequences and labelled score sequences: results of a computer enumeration

$s(n)$ and $p(n, n(n-1), r)$ have been computed for $n \leqslant 22$ and $n - 1 \leqslant r \leqslant 2n - 2$, and $ls(n)$ and $lp(n, n(n-1), r)$ for $n \leqslant 12$ and $n - 1 \leqslant r \leqslant 2n - 2$. For $n \leqslant 10$, $p(n, n(n-1), r)$ and $lp(n, n(n-1), r)$ are tabulated with $s(n)$ and $ls(n)$ as the respective totals. For $n > 10$, only $s(n)$ and $ls(n)$ are listed.

	Greatest score	Number of score sequences	Number of labelled score sequences
n	r	p(n,n(n-1),r)	lp(n,n(n-1),r)
1	0	1	1
2	1	1	1
	2	1	2
	Total	2	3
3	2	1	1
	3	2	9
	4	2	9
	Total	5	19
4	3	1	1
	4	4	34
	5	6	90

n	r	p(n,n(n-1),r)	lp(n,n(n-1),r)
4	6	5	76
	Total	16	201
5	4	1	1
	5	6	125
	6	16	700
	7	20	1,250
	8	16	1,005
	Total	59	3,081
6	5	1	1
	6	10	461
	7	35	4,970
	8	65	15,470
	9	77	23,295
	10	59	18,486
	Total	247	62,683
7	6	1	1
	7	14	1,715
	8	71	33,810
	9	174	170,324
	10	283	400,869
	11	321	553,455
	12	247	438,781
	Total	1,111	1,598,955
8	7	1	1
	8	21	6,434

n	r	p(n,n(n-1),r)	lp(n,n(n-1),r)
8	9	134	224,994
	10	425	1,761,354
	11	868	6,076,938
	12	1,296	12,236,476
	13	1,446	16,082,276
	14	1,111	12,791,640
	Total	5,302	49,180,113
9	8	1	1
	9	29	24,309
	10	243	1,480,050
	11	950	17,541,414
	12	2,396	85,828,572
	13	4,375	237,553,530
	14	6,224	434,418,240
	15	6,856	553,938,516
	16	5,302	442,621,017
	Total	26,376	1,773,405,649
10	9	1	1
	10	41	92,377
	11	421	9,679,527
	12	2,018	170,543,802
	13	6,071	1,159,080,417
	14	13,250	4,284,437,772
	15	22,545	10,298,178,720
	16	31,011	17,667,086,580
	17	33,936	22,087,513,485
	18	26,376	17,734,056,490
	Total	135,670	73,410,669,171

n	s(n)	ls(n)
11	716,542	3,432,267,261,699
12	3,868,142	178,922,825,114,905
13	21,265,884	
14	118,741,369	
15	671,906,876	
16	3,846,342,253	
17	22,243,294,360	
18	129,793,088,770	
19	763,444,949,789	
20	4,522,896,682,789	
21	26,968,749,517,543	
22	161,750,625,450,884	

Table 2

The number of score structures and labelled score structures: results of a computer enumeration

$p(n, \frac{1}{2}n(n-1), r)$ is the number of score structures $(l_1, l_2, \ldots, l_n)$ with $l_n = r$ and $lp(n, \frac{1}{2}n(n-1), r)$ is the number of labelled score structures, defined analogously to labelled score sequences, with $r = \max_{k\in\{1,2,\ldots,n\}}\{l_k\}$. Here $\left[\frac{n+1}{2}\right] \leq r \leq n - 1$. The respective totals are $l(n)$ and $ll(n)$, the numbers of unlabelled and labelled score structures. For $n > 10$, only $l(n)$ and $ll(n)$ are listed. The values of $l(n)$ for $n = 1, \ldots, 36$ first appeared in [34].

n	Greatest score r	Number of score sequences $p(n,\frac{1}{2}n(n-1),r)$	Number of labelled score sequences $lp(n,\frac{1}{2}n(n-1),r)$
1	0	1	1
2	1	1	2
3	1	1	1
	2	1	6
	Total	2	7
4	2	2	10
	3	2	28
	Total	4	38
5	2	1	1
	3	4	100
	4	4	190
	Total	9	291

n	r	$p(n,\frac{1}{2}n(n-1),r)$	$lp(n,\frac{1}{2}n(n-1),r)$
6	3	3	56
	4	10	1,130
	5	9	1,746
	Total	22	2,932
7	3	1	1
	4	10	1,519
	5	26	14,917
	6	22	20,524
	Total	59	36,961
8	4	5	330
	5	33	35,770
	6	70	230,160
	7	59	295,688
	Total	167	561,948
9	4	1	1
	5	22	22,824
	6	103	838,644
	7	197	4,107,504
	8	167	5,057,532
	Total	490	10,026,505
10	5	7	2,002
	6	88	1,057,452
	7	321	20,591,742
	8	580	83,692,290

n	r	p(n,½n(n-1),r)	lp(n,½n(n-1),r)
10	9	490	100,265,050
	Total	1,486	205,608,536

n	l(n)	ll(n)
11	4,639	4,767,440,679
12	14,805	123,373,203,208
13	48,107	3,525,630,110,107
14	158,808	110,284,283,006,640
15	531,469	
16	1,799,659	
17	6,157,068	
18	21,258,104	
19	73,996,100	
20	259,451,116	
21	915,695,102	
22	3,251,073,303	
23	11,605,141,649	
24	41,631,194,766	
25	150,021,775,417	
26	542,875,459,724	
27	1,972,050,156,181	
28	7,189,259,574,618	
29	26,295,934,251,565	
30	96,478,910,768,821	

It is noticeable from the results of the computer enumeration that for score sequences of order n when $4 \leqslant n \leqslant 22$,

$$p(n, n(n-1), r) < p(n, n(n-1), r + 1) \quad \text{for } n - 1 \leqslant r \leqslant 2n - 4$$

and $$p(n, n(n-1), 2n - 3) > p(n, n(n-1), 2n - 2),$$

while the same pattern is apparent for labelled score sequences of order n when $4 \leqslant n \leqslant 12$.

For score structures (see Table 2 for the definitions) of order n when $6 \leqslant n \leqslant 30$,

$$p(n, \tfrac{1}{2}n(n-1), r) < p(n, \tfrac{1}{2}n(n-1), r + 1) \quad \text{for } \left[\tfrac{n+1}{2}\right] \leqslant r \leqslant n - 3$$

and $$p(n, \tfrac{1}{2}n(n-1), n - 2) > p(n, \tfrac{1}{2}n(n-1), n - 1),$$

but for labelled score structures of order n when $3 \leqslant n \leqslant 14$,

$$lp(n, \tfrac{1}{2}n(n-1), r) < lp(n, \tfrac{1}{2}n(n-1), r + 1) \quad \text{for } \left[\tfrac{n+1}{2}\right] \leqslant r \leqslant n - 2.$$

These observations are made from data available for small values of n only, and may not be indicative of any general trends.

APPENDIX 2

The number of semiorders and labelled semiorders

$ld(n)$ and $lc(n, r)$ have been computed for $n \leqslant 11$ and $1 \leqslant r \leqslant n$. These values of $ld(n)$ and $lc(n, r)$ are tabulated below together with the corresponding values of $d(n) = \frac{1}{n+1}\binom{2n}{n}$ and $c(n, r)$. The values of $c(n, r)$ were calculated recursively by using Proposition (3.3.4). The greatest score s_n is related to r by $s_n = n + r - 2$.

		Greatest score	Number of semiorders	Number of labelled semiorders
n	r	s_n	$c(n,r)$	$lc(n,r)$
1	1	0	1	1
2	1	1	1	1
	2	2	1	2
		Total	2	3
3	1	2	1	1
	2	3	2	9
	3	4	2	9
		Total	5	19
4	1	3	1	1
	2	4	3	28
	3	5	5	78
	4	6	5	76
		Total	14	183

n	r	s_n	c(n,r)	lc(n,r)
5	1	4	1	1
	2	5	4	75
	3	6	9	450
	4	7	14	930
	5	8	14	915
		Total	42	2,371
6	1	5	1	1
	2	6	5	186
	3	7	14	2,175
	4	8	28	7,700
	5	9	42	14,415
	6	10	42	14,226
		Total	132	38,703
7	1	6	1	1
	2	7	6	441
	3	8	20	9,513
	4	9	48	54,075
	5	10	90	154,035
	6	11	132	274,113
	7	12	132	270,921
		Total	429	763,099
8	1	7	1	1
	2	8	7	1,016
	3	9	27	39,004
	4	10	75	345,464

n	r	s_n	c(n,r)	lc(n,r)
8	5	11	165	1,407,070
	6	12	297	3,580,472
	7	13	429	6,171,004
	8	14	429	6,104,792
		Total	1,430	17,648,823
9	1	8	1	1
	2	9	8	2,295
	3	10	35	152,820
	4	11	110	2,069,172
	5	12	275	11,782,638
	6	13	572	40,111,470
	7	14	1,001	95,184,180
	8	15	1,430	160,461,108
	9	16	1,430	158,839,407
		Total	4,862	468,603,091
10	1	9	1	1
	2	10	9	5,110
	3	11	44	578,925
	4	12	154	11,811,000
	5	13	429	93,026,010
	6	14	1,001	413,834,652
	7	15	2,002	1,261,051,050
	8	16	3,432	2,852,787,000
	9	17	4,862	4,731,717,645
	10	18	4,862	4,686,030,910
		Total	16,796	14,050,842,303

n	r	s_n	c(n,r)	lc(n,r)
11	1	10	1	1
	2	11	10	11,253
	3	12	54	2,136,915
	4	13	208	64,921,725
	5	14	637	702,332,730
	6	15	1,638	4,044,307,806
	7	16	3,640	15,425,973,486
	8	17	7,072	43,593,100,650
	9	18	11,934	95,239,122,165
	10	19	16,796	156,012,323,115
	11	20	16,796	154,559,265,333
		Total	58,786	469,643,495,179

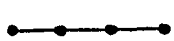
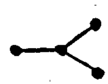
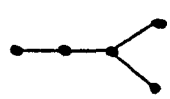
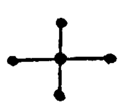
It is noticeable from the results of the computer enumeration that when $4 \leqslant n \leqslant 11$, $lc(n, r) < lc(n, r + 1)$ for $1 \leqslant r < n - 1$ and $lc(n, n - 1) > lc(n, n)$.

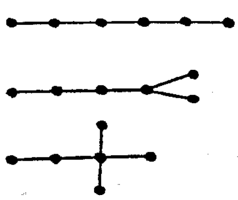
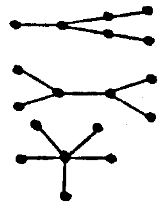
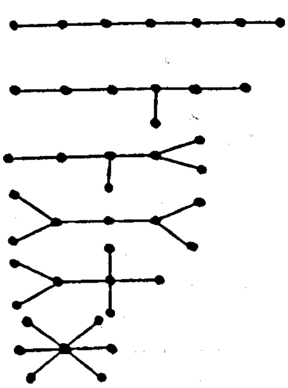
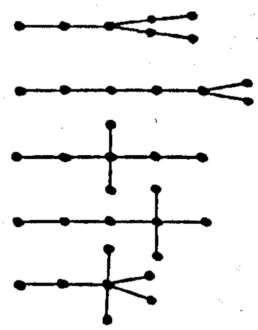
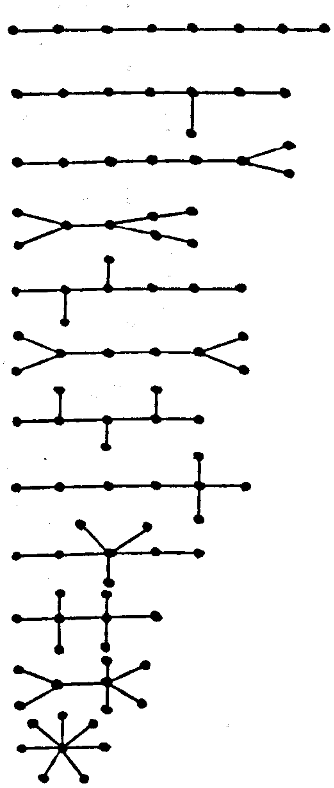
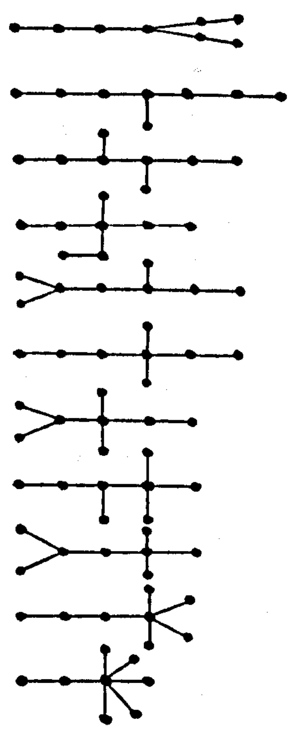
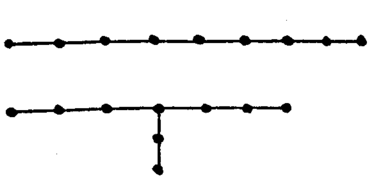
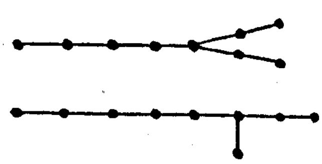
Also, it follows from Proposition (3.3.4) that $c(n, r) < c(n, r + 1)$ for $1 \leqslant r < n - 1$ and $c(n, n - 1) = c(n, n)$.

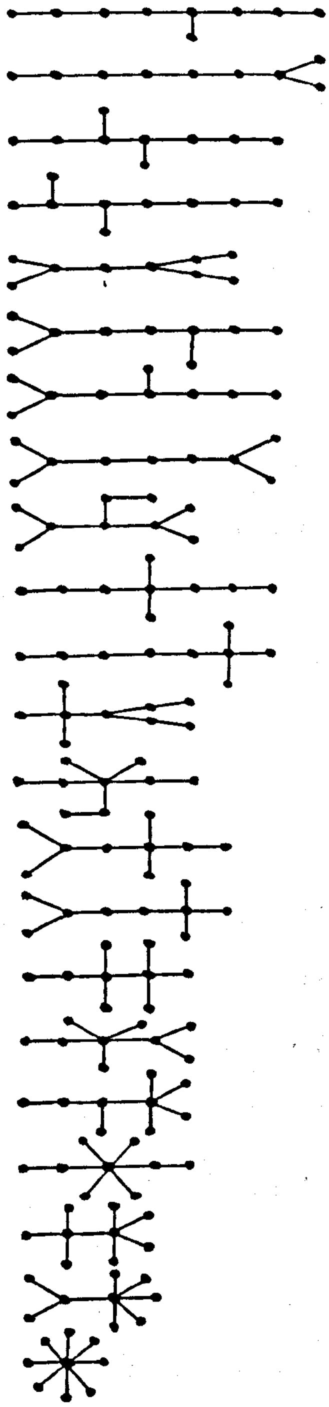
APPENDIX 3

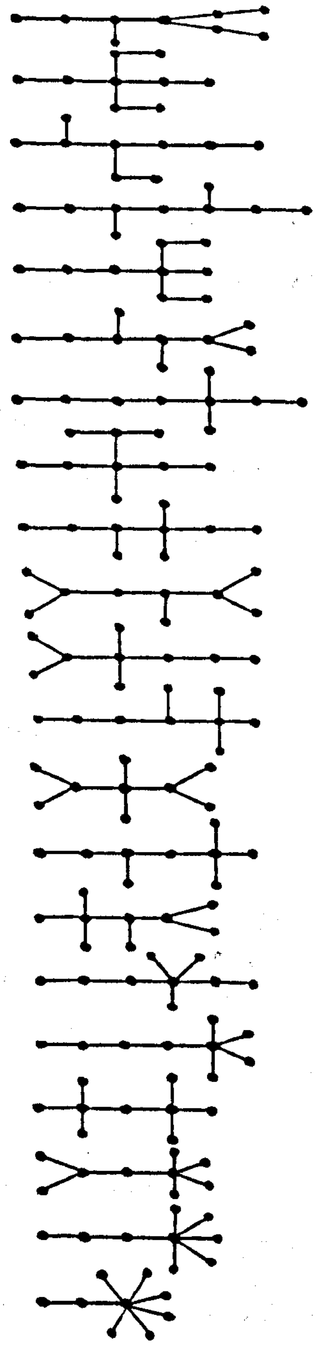
Tree diagrams and the number of open sets of maximal connected topologies.

The cardinalities of all maximal connected topologies of order $n \leqslant 9$ are determined here by presenting a list of tree diagrams together with the cardinalities of their associated topologies. Trees of a given order are listed in order of increasing cardinality. The diagrams are taken from [18]. The cardinality of each tree was evaluated by using Corollary (4.4.12) and the fact that if an acyclic graph F (called a *forest*) has components $T_1, \ldots, T_k$ (which are all trees) then $|F| = |T_1| \ldots |T_k|$.

n		Cardinality		Cardinality
1		2		
2		3		
3		5		
4		8		9
5		13		14
		17		

n		Cardinality		Cardinality
6		21 23 26		22 24 33
7		34 36 38 41 44 65		35 37 40 43 50
8		55 58 60 61 62 65 66 69 76 80 84 129		57 59 60 62 64 66 68 70 77 83 98
9		89 93		92 94





n	Cardinality	Cardinality
9	95	96
	97	97
	98	99
	100	100
	101	102
	102	104
	105	106
	106	106
	107	108
	109	110
	112	112
	113	114
	116	116
	118	120
	121	122
	124	126
	128	133
	134	145
	148	149
	152	163
	164	194
	257	

APPENDIX 4

Partial order diagrams

This appendix contains a list of diagrams of all the partial orders with $n \leq 6$ vertices. For each value of n, the partial orders with n vertices have been ordered lexicographically by score sequence from the least value of the n-tuple $s_1\ s_2\ \dots\ s_n$ to the greatest value. The index number of each partial order is written in the top-left corner, and the score sequence is written directly below the diagram. Where there is more than one partial order with the same score sequence S, then $\bar{D}(S)$, which by Theorem (1.5.2) has the minimum number of arcs, is listed first. The number of arcs is written in the bottom-left corner.

The labelling of each partial order is its *canonical labelling.* If the n x n adjacency matrix $A = (a_{ij})$ of an oriented graph is defined by $a_{ij} = 1$ if iRj and $a_{ij} = 0$ otherwise, then its canonical labelling is that which maximises the n^2-tuple

$$a_{11}\ \dots\ a_{1n}\ a_{21}\ \dots\ a_{2n}\ \dots\ a_{n1}\ \dots\ a_{nn}.$$

The canonical labelling is unique since any two labellings which both maximise $a_{11}\ \dots\ a_{nn}$ are isomorphic. For each n, this labelling is given in the first diagram only, the labellings of the remaining diagrams being identical: with vertex 1 at the apex and the labels increasing in a clockwise direction. As usual the score sequence is written in non-decreasing order, although $s_1 \leq s_2 \leq \dots \leq s_n$ for the canonical labelling only if all the vertex scores of the partial order are equal. The total number of non-isomorphic labellings of the partial order is written in the top-right corner.

An automorphism of a labelled oriented graph is a permutation of the vertices which preserves adjacency. The set of all automorphisms of a labelled oriented graph with n vertices forms a permutation group of degree n. This *automorphism group* is written in the bottom-right corner. The notation is as follows:

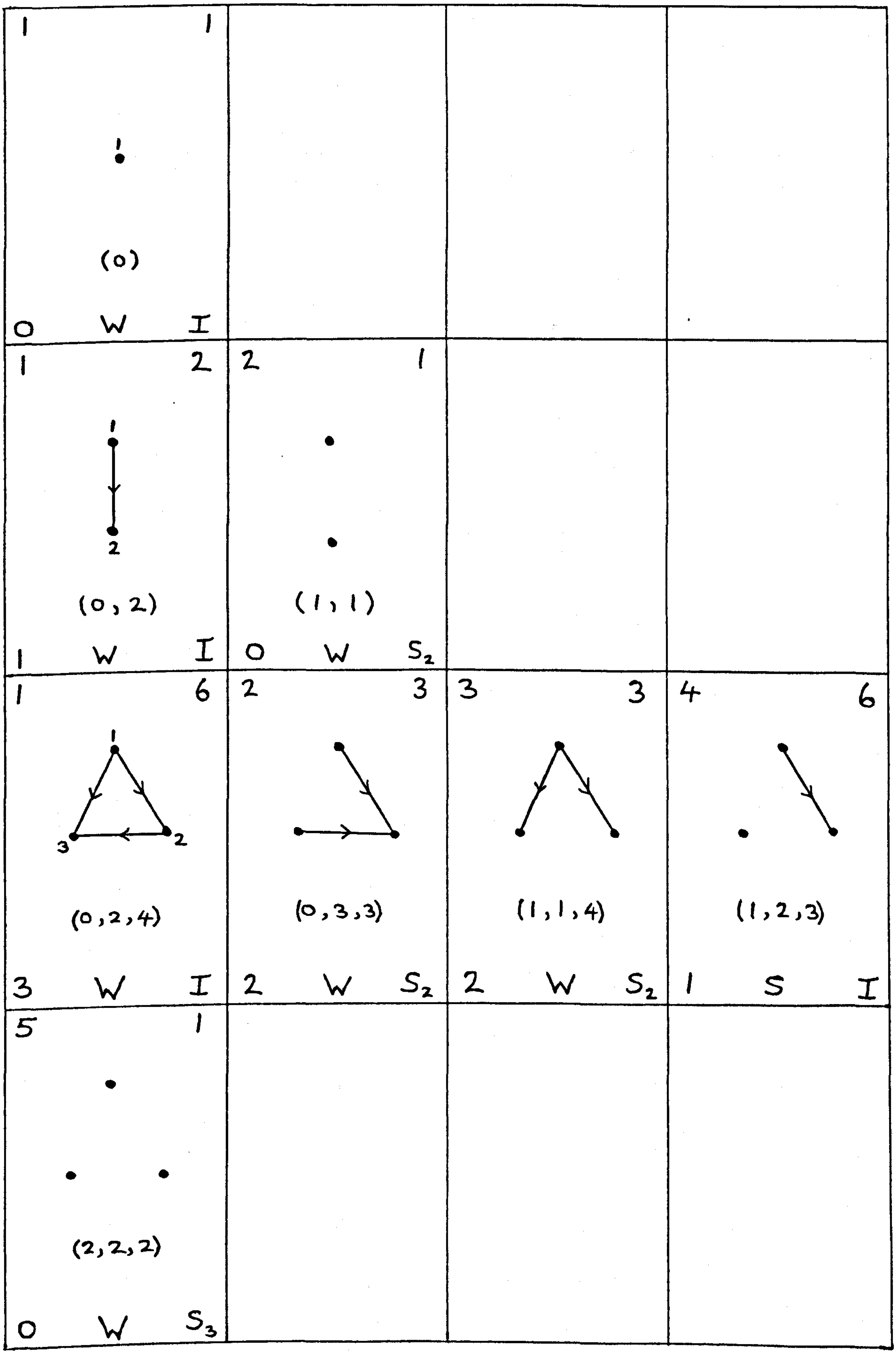
I is the identity group (of order 1),

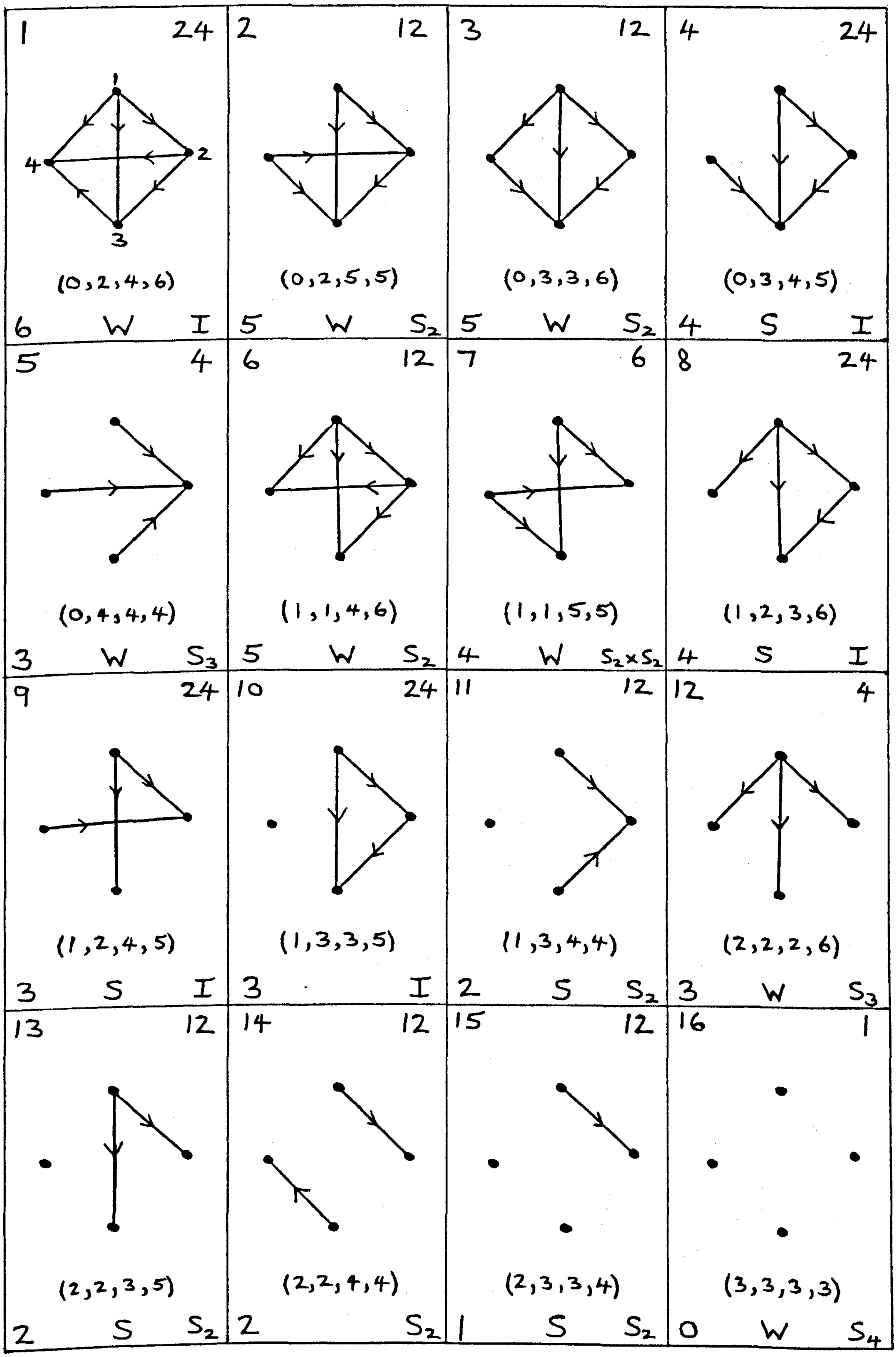
S_r is the symmetric group of degree r (of order r!),

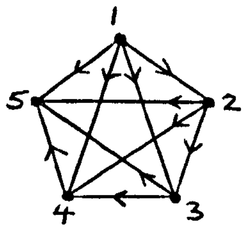
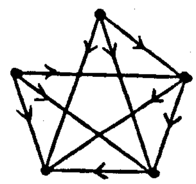
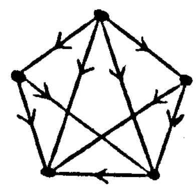
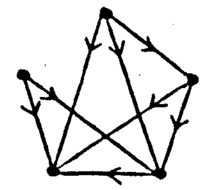
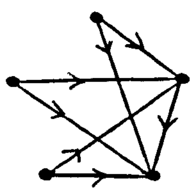
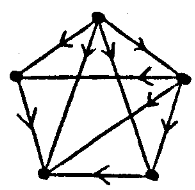
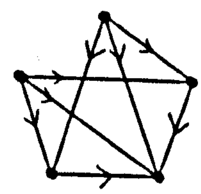
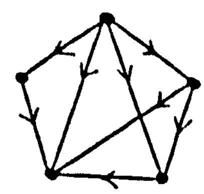
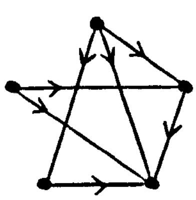
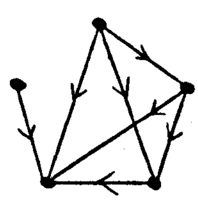
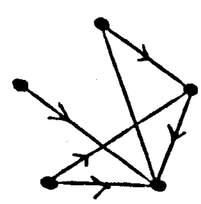
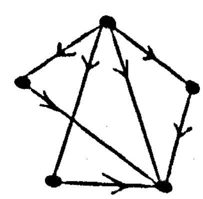
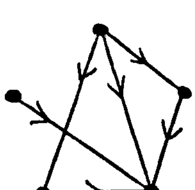
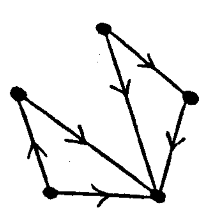
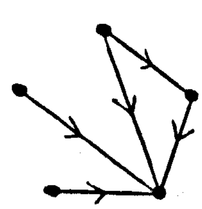
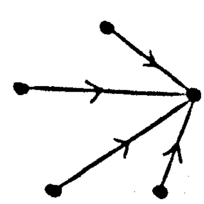
D_8 is the dihedral group of order 8,

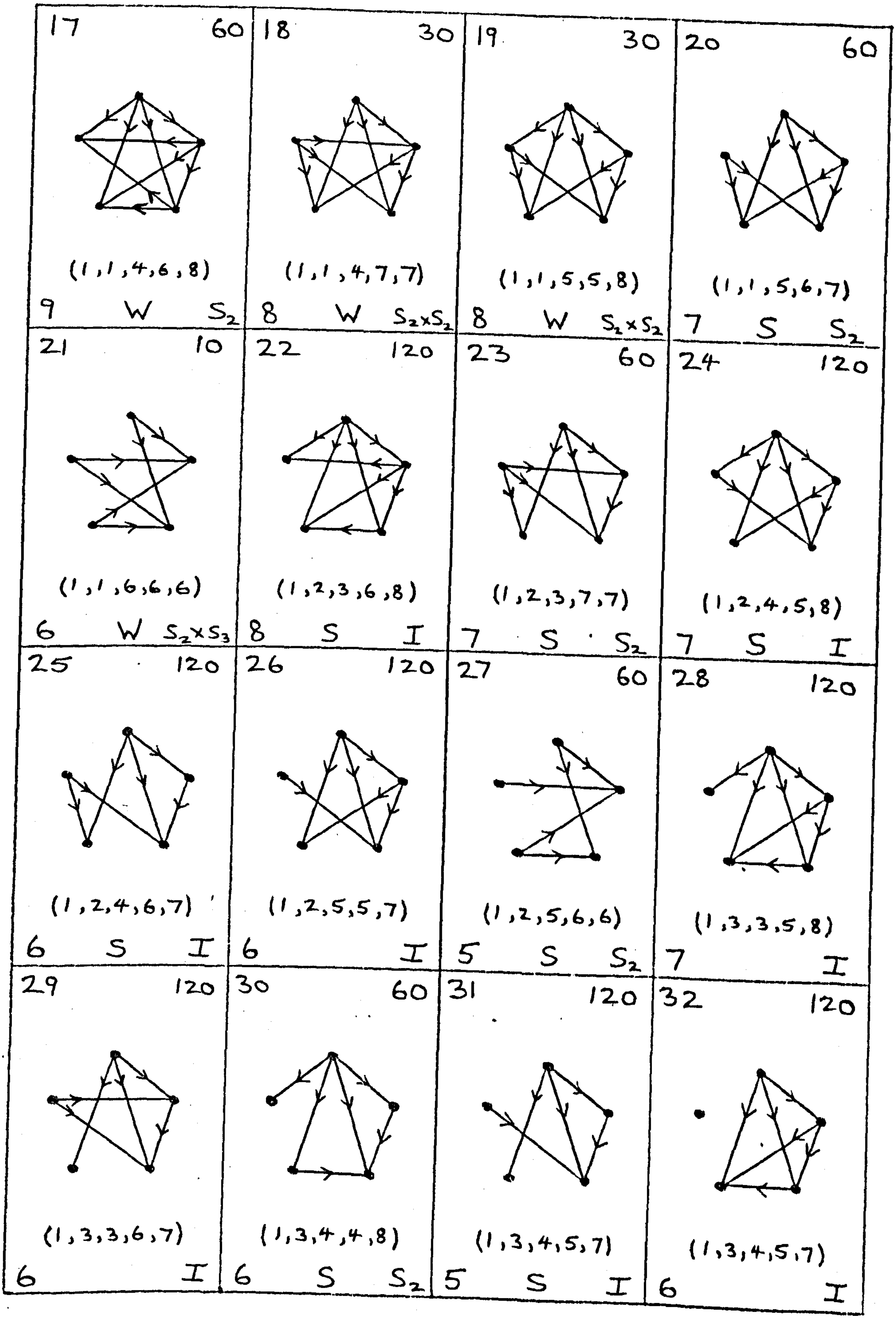
x indicates the direct product of groups.

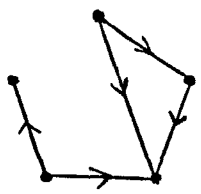
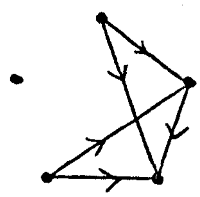
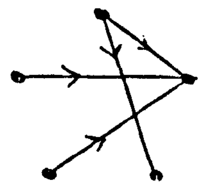
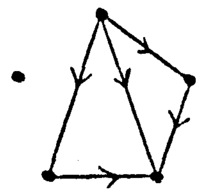
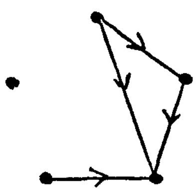
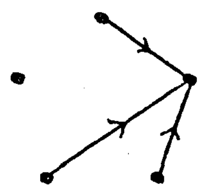
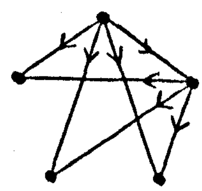
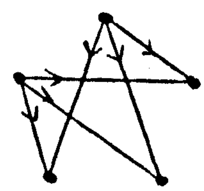
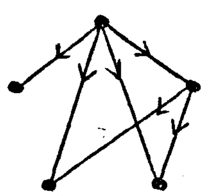
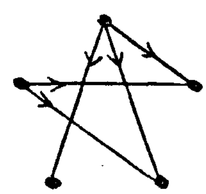
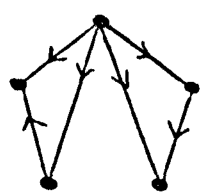
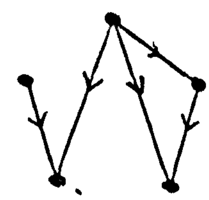
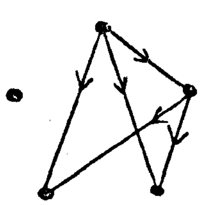
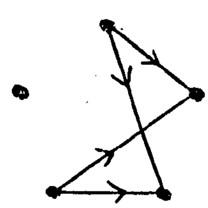
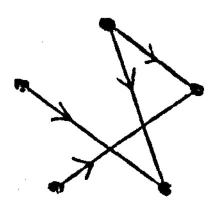
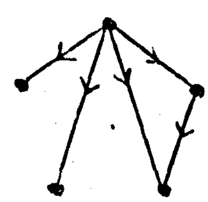
Those partial orders which are weak orders are indicated by a letter W in the bottom line, and those which are semiorders but not weak orders are indicated by S. Notice that for given n and s_1, where $0 \leq s_1 < n - 1$, the number of weak orders with least score s_1 is 2^{n+s_1-2} and the number of semiorders with least score s_1 is $c(n, n - s_1)$.


1 1
1
(0)
0 W I
1 2
1
2
(0, 2)
1 W I
2 1
(1, 1)
0 W S_2
1 6
1
3
2
(0, 2, 4)
3 W I
2 3
(0, 3, 3)
2 W S_2
3 3
(1, 1, 4)
2 W S_2
4 6
(1, 2, 3)
1 S I
5 1
(2, 2, 2)
0 W S_3

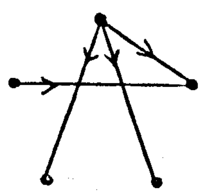

1 24
1
2
3
4
(0,2,4,6)
6 W I
2 12
(0,2,5,5)
5 W S_2
3 12
(0,3,3,6)
5 W S_2
4 24
(0,3,4,5)
4 S I
5 4
(0,4,4,4)
3 W S_3
6 12
(1,1,4,6)
5 W S_2
7 6
(1,1,5,5)
4 W $S_2 \times S_2$
8 24
(1,2,3,6)
4 S I
9 24
(1,2,4,5)
3 S I
10 24
(1,3,3,5)
3 I
11 12
(1,3,4,4)
2 S S_2
12 4
(2,2,2,6)
3 W S_3
13 12
(2,2,3,5)
2 S S_2
14 12
(2,2,4,4)
2 S_2
15 12
(2,3,3,4)
1 S S_2
16 1
(3,3,3,3)
0 W S_4

1 120  (0,2,4,6,8) 10 W I	2 60  (0,2,4,7,7) 9 W S_2	3 60  (0,2,5,5,8) 9 W S_2	4 120  (0,2,5,6,7) 8 S I
5 20  (0,2,6,6,6) 7 W S_3	6 60  (0,3,3,6,8) 9 W S_2	7 30  (0,3,3,7,7) 8 W $S_2 \times S_2$	8 120  (0,3,4,5,8) 8 S I
9 120  (0,3,4,6,7) 7 S I	10 120  (0,3,5,5,7) 7 I	11 60  (0,3,5,6,6) 6 S S_2	12 20  (0,4,4,4,8) 7 W S_3
13 60  (0,4,4,5,7) 6 S S_2	14 60  (0,4,4,6,6) 6 S_2	15 60  (0,4,5,5,6) 5 S S_2	16 5  (0,5,5,5,5) 4 W S_4


17 60
(1,1,4,6,8)
9 W S_2
18 30
(1,1,4,7,7)
8 W $S_2 \times S_2$
19 30
(1,1,5,5,8)
8 W $S_2 \times S_2$
20 60
(1,1,5,6,7)
7 S S_2
21 10
(1,1,6,6,6)
6 W $S_2 \times S_3$
22 120
(1,2,3,6,8)
8 S I
23 60
(1,2,3,7,7)
7 S S_2
24 120
(1,2,4,5,8)
7 S I
25 120
(1,2,4,6,7)
6 S I
26 120
(1,2,5,5,7)
6 I
27 60
(1,2,5,6,6)
5 S S_2
28 120
(1,3,3,5,8)
7 I
29 120
(1,3,3,6,7)
6 I
30 60
(1,3,4,4,8)
6 S S_2
31 120
(1,3,4,5,7)
5 S I
32 120
(1,3,4,5,7)
6 I

33 120  (1,3,4,6,6) 5 I	34 60  (1,3,4,6,6) 5 S_2	35 60  (1,3,5,5,6) 4 S S_2	36 60  (1,4,4,4,7) 5 S_2
37 120  (1,4,4,5,6) 4 I	38 20  (1,4,5,5,5) 3 S S_3	39 20  (2,2,2,6,8) 7 W S_3	40 10  (2,2,2,7,7) 6 W $S_2 \times S_3$
41 60  (2,2,3,5,8) 6 S S_2	42 60  (2,2,3,6,7) 5 S S_2	43 60  (2,2,4,4,8) 6 S_2	44 120  (2,2,4,5,7) 5 I
45 60  (2,2,4,5,7) 5 S_2	46 30  (2,2,4,6,6) 4 S $S_2 \times S_2$	47 60  (2,2,5,5,6) 4 S_2	48 60  (2,3,3,4,8) 5 S S_2

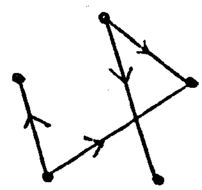
49 60



(2,3,3,5,7)

4 S S_2

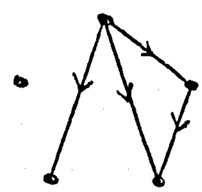
50 60



(2,3,3,6,6)

4 S_2

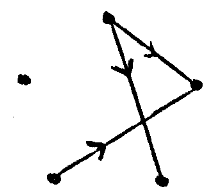
51 120



(2,3,4,4,7)

4 I

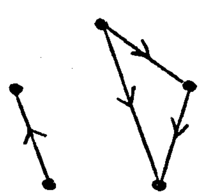
52 120



(2,3,4,5,6)

3 S I

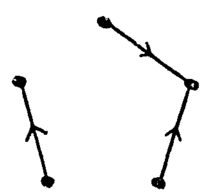
53 120



(2,3,4,5,6)

4 I

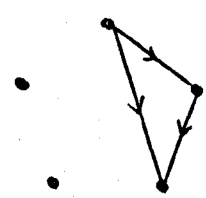
54 60



(2,3,5,5,5)

3 S_2

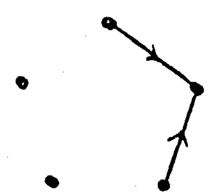
55 60



(2,4,4,4,6)

3 S_2

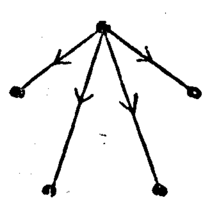
56 30



(2,4,4,5,5)

2 S $S_2 \times S_2$

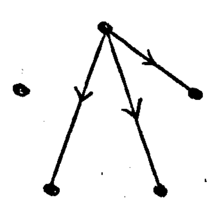
57 5



(3,3,3,3,8)

4 W S_4

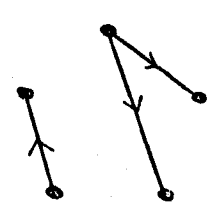
58 20



(3,3,3,4,7)

3 S S_3

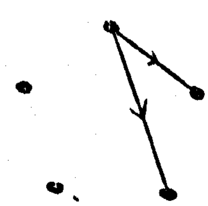
59 60



(3,3,3,5,6)

3 S_2

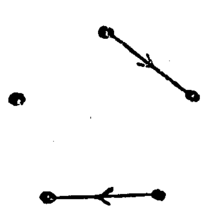
60 30



(3,3,4,4,6)

2 S $S_2 \times S_2$

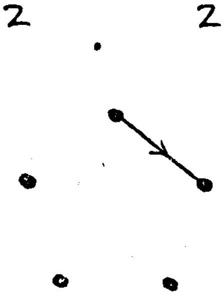
61 60



(3,3,4,5,5)

2 S_2

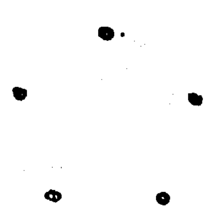
62 20



(3,4,4,4,5)

1 S S_3

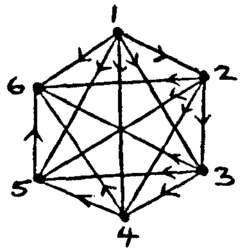
63 1



(4,4,4,4,4)

0 W S_5

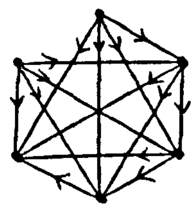
1 720



(0,2,4,6,8,10)

15 W I

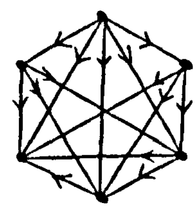
2 360



(0,2,4,6,9,9)

14 W S_2

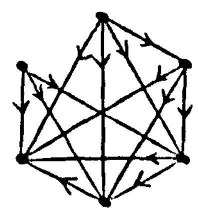
3 360



(0,2,4,7,7,10)

14 W S_2

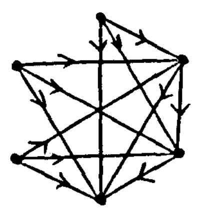
4 720



(0,2,4,7,8,9)

13 S I

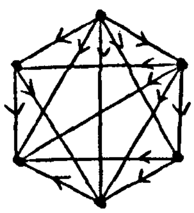
5 120



(0,2,4,8,8,8)

12 W S_3

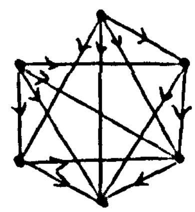
6 360



(0,2,5,5,8,10)

14 W S_2

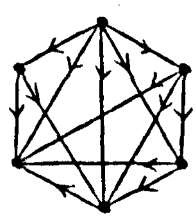
7 180



(0,2,5,5,9,9)

13 W $S_2 \times S_2$

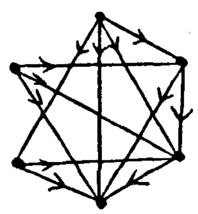
8 720



(0,2,5,6,7,10)

13 S I

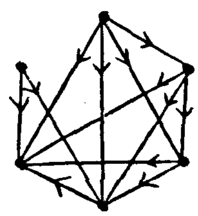
9 720



(0,2,5,6,8,9)

12 S I

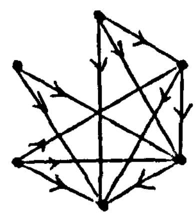
10 720



(0,2,5,7,7,9)

12 I

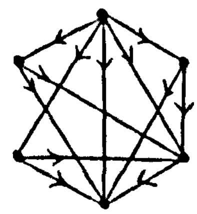
11 360



(0,2,5,7,8,8)

11 S S_2

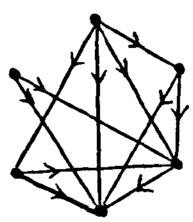
12 120



(0,2,6,6,6,10)

12 W S_3

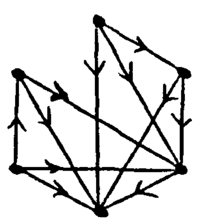
13 360



(0,2,6,6,7,9)

11 S S_2

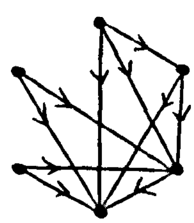
14 360



(0,2,6,6,8,8)

11 S_2

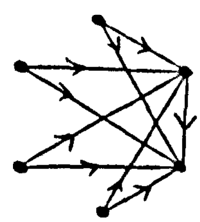
15 360



(0,2,6,7,7,8)

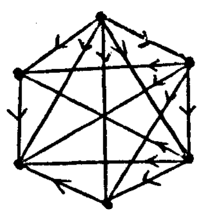
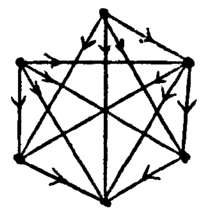
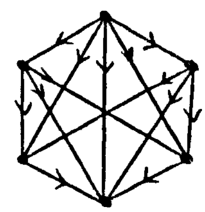
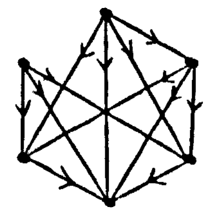
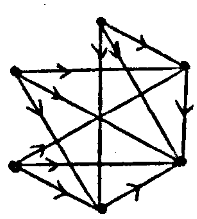
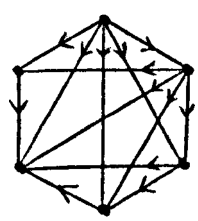
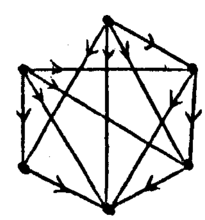
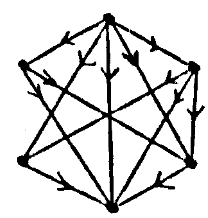
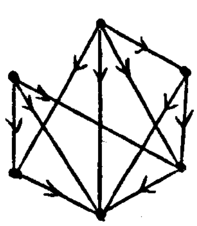
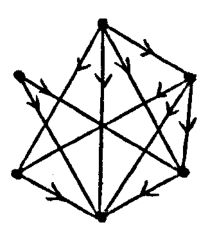
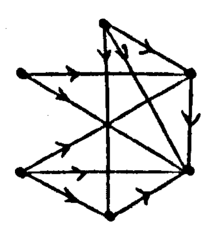
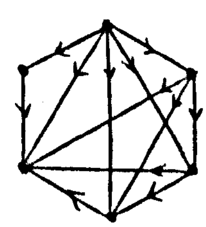
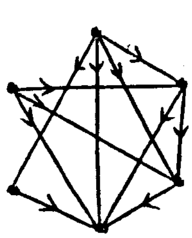
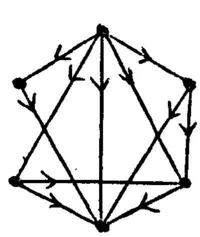
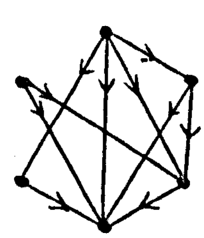
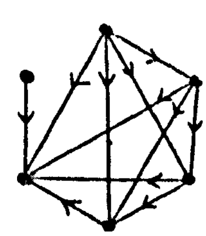
10 S S_2

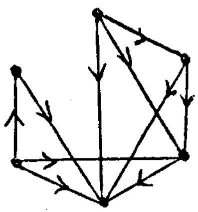
16 30

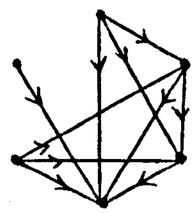


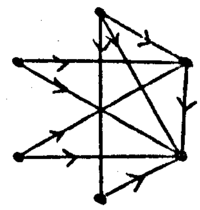
(0,2,7,7,7,7)

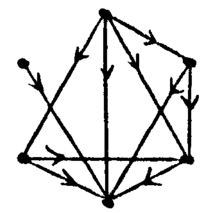
9 W S_4

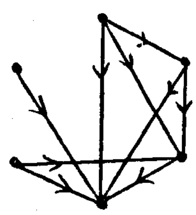
17 360  (0,3,3,6,8,10) 14 W S_2	18 180  (0,3,3,6,9,9) 13 W $S_2 \times S_2$	19 180  (0,3,3,7,7,10) 13 W $S_2 \times S_2$	20 360  (0,3,3,7,8,9) 12 S S_2
21 60  (0,3,3,8,8,8) 11 W $S_2 \times S_3$	22 720  (0,3,4,5,8,10) 13 S I	23 360  (0,3,4,5,9,9) 12 S S_2	24 720  (0,3,4,6,7,10) 12 S I
25 720  (0,3,4,6,8,9) 11 S I	26 720  (0,3,4,7,7,9) 11 I	27 360  (0,3,4,7,8,8) 10 S S_2	28 720  (0,3,5,5,7,10) 12 I
29 720  (0,3,5,5,8,9) 11 I	30 360  (0,3,5,6,6,10) 11 S S_2	31 720  (0,3,5,6,7,9) 10 S I	32 720  (0,3,5,6,7,9) 11 I

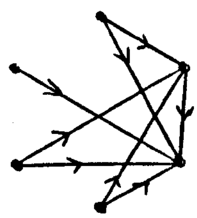
33 720

(0,3,5,6,8,8)
10 I

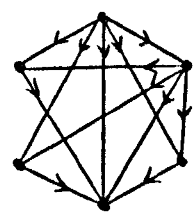
34 360

(0,3,5,6,8,8)
10 S_2

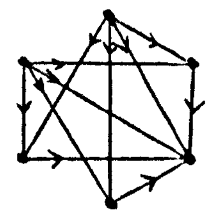
35 360

(0,3,5,7,7,8)
9 S S_2

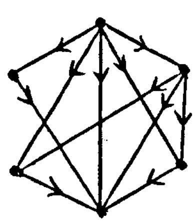
36 360

(0,3,6,6,6,9)
10 S_2

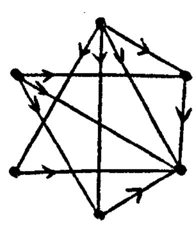
37 720

(0,3,6,6,7,8)
9 I

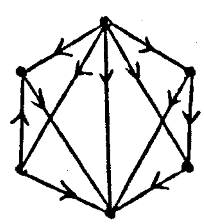
38 120

(0,3,6,7,7,7)
8 S S_3

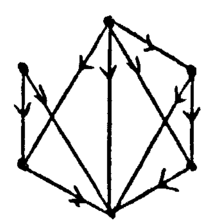
39 120

(0,4,4,4,8,10)
12 W S_3

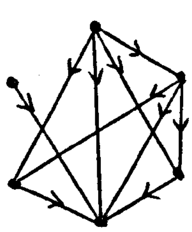
40 60

(0,4,4,4,9,9)
11 W $S_2 \times S_3$

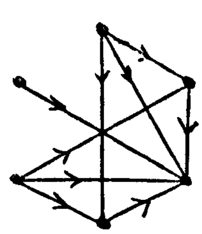
41 360

(0,4,4,5,7,10)
11 S S_2

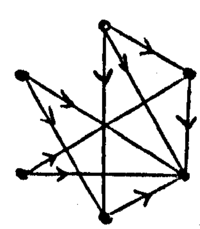
42 360

(0,4,4,5,8,9)
10 S S_2

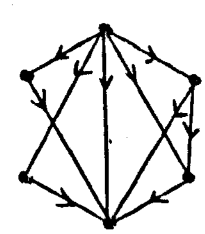
43 360

(0,4,4,6,6,10)
11 S_2

44 720

(0,4,4,6,7,9)
10 I

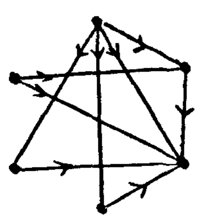
45 360

(0,4,4,6,7,9)
10 S_2

46 180

(0,4,4,6,8,8)
9 S $S_2 \times S_2$

47 360

(0,4,4,7,7,8)
9 S_2

48 360

(0,4,5,5,6,10)
10 S S_2

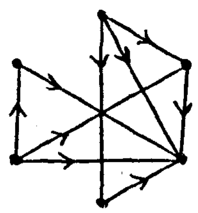
49 360



(0, 4, 5, 5, 7, 9)

9 S S_2

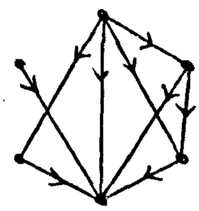
50 360



(0, 4, 5, 5, 8, 8)

9 S_2

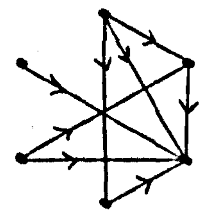
51 720



(0, 4, 5, 6, 6, 9)

9 I

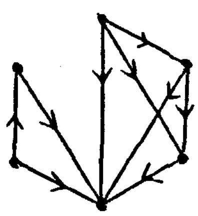
52 720



(0, 4, 5, 6, 7, 8)

8 S I

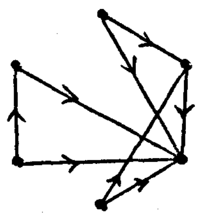
53 720



(0, 4, 5, 6, 7, 8)

9 I

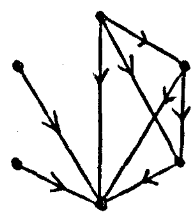
54 360



(0, 4, 5, 7, 7, 7)

9 S_2

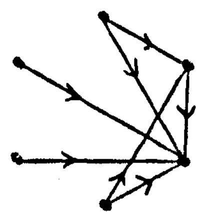
55 360



(0, 4, 6, 6, 6, 8)

8 S_2

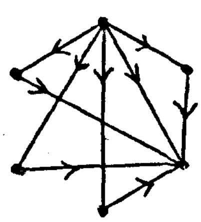
56 180



(0, 4, 6, 6, 7, 7)

7 S $S_2 \times S_2$

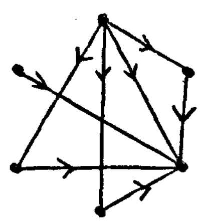
57 30



(0, 5, 5, 5, 5, 10)

9 W S_4

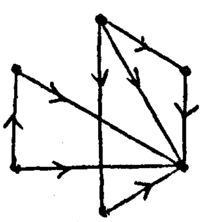
58 120



(0, 5, 5, 5, 6, 9)

8 S S_3

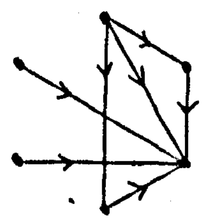
59 360



(0, 5, 5, 5, 7, 8)

8 S_2

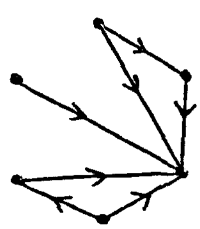
60 180



(0, 5, 5, 6, 6, 8)

7 S $S_2 \times S_2$

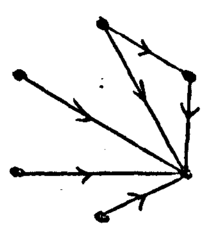
61 360



(0, 5, 5, 6, 7, 7)

7 S_2

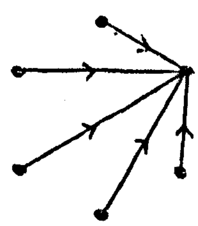
62 120



(0, 5, 6, 6, 6, 7)

6 S S_3

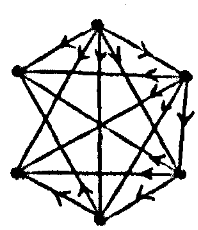
63 6



(0, 6, 6, 6, 6, 6)

5 W S_5

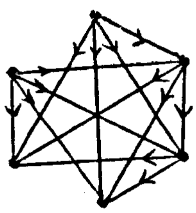
64 360



(1, 1, 4, 6, 8, 10)

14 W S_2

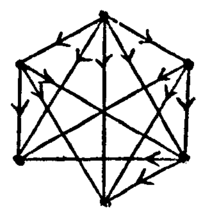
65 180



(1,1,4,6,9,9)

13 W $S_2 \times S_2$

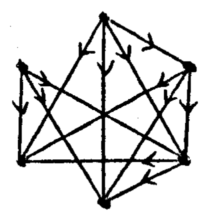
66 180



(1,1,4,7,7,10)

13 W $S_2 \times S_2$

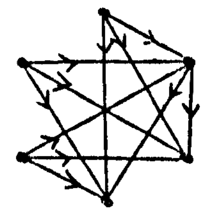
67 360



(1,1,4,7,8,9)

12 S S_2

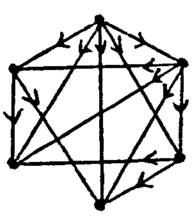
68 60



(1,1,4,8,8,8)

11 W $S_2 \times S_3$

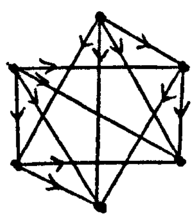
69 180



(1,1,5,5,8,10)

13 W $S_2 \times S_2$

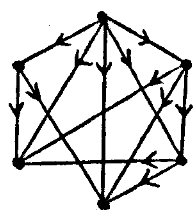
70 90



(1,1,5,5,9,9)

12 W $S_2 \times S_2 \times S_2$

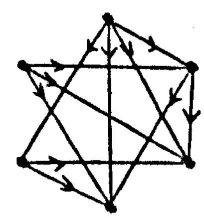
71 360



(1,1,5,6,7,10)

12 S S_2

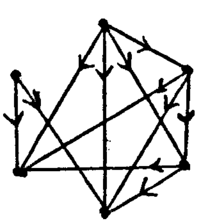
72 360



(1,1,5,6,8,9)

11 S S_2

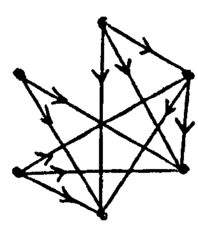
73 360



(1,1,5,7,7,9)

11 S_2

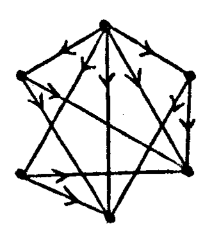
74 180



(1,1,5,7,8,8)

10 S $S_2 \times S_2$

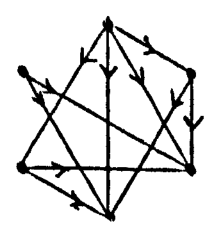
75 60



(1,1,6,6,6,10)

11 W $S_2 \times S_3$

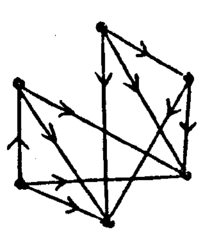
76 180



(1,1,6,6,7,9)

10 S $S_2 \times S_2$

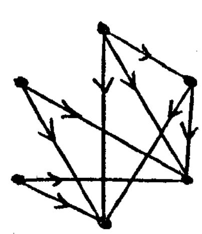
77 180



(1,1,6,6,8,8)

10 $S_2 \times S_2$

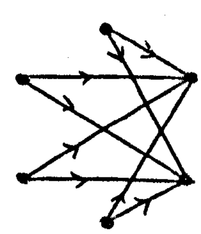
78 180



(1,1,6,7,7,8)

9 S $S_2 \times S_2$

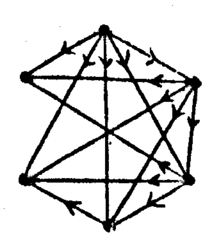
79 15



(1,1,7,7,7,7)

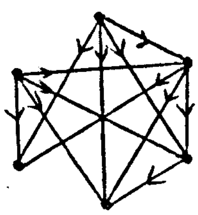
8 W $S_2 \times S_4$

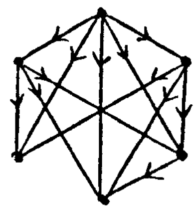
80 720

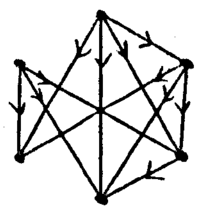


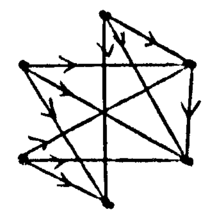
(1,2,3,6,8,10)

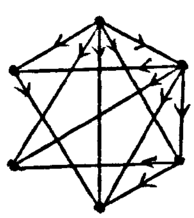
13 S I

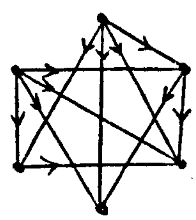
81 360

(1,2,3,6,9,9)
12 S S_2

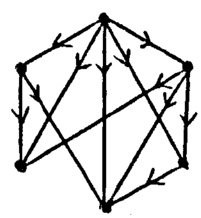
82 360

(1,2,3,7,7,10)
12 S S_2

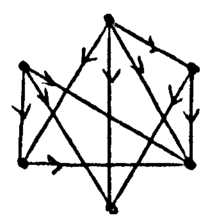
83 720

(1,2,3,7,8,9)
11 S I

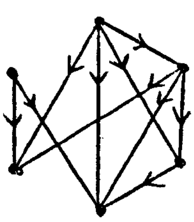
84 120

(1,2,3,8,8,8)
10 S S_3

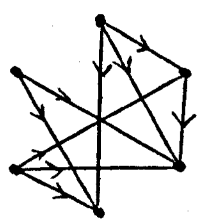
85 720

(1,2,4,5,8,10)
12 S I

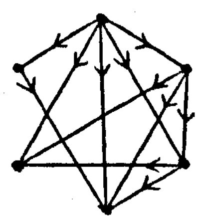
86 360

(1,2,4,5,9,9)
11 S S_2

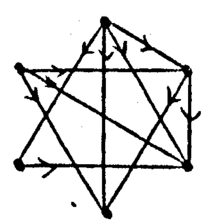
87 720

(1,2,4,6,7,10)
11 S I

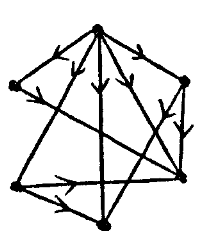
88 720

(1,2,4,6,8,9)
10 S I

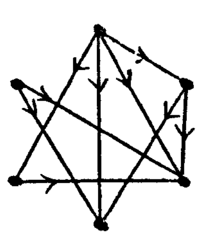
89 720

(1,2,4,7,7,9)
10 I

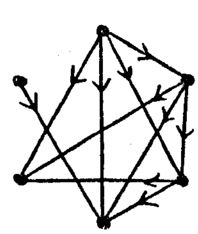
90 360

(1,2,4,7,8,8)
9 S S_2

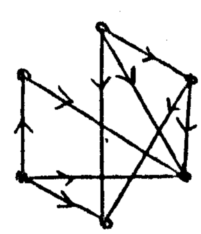
91 720

(1,2,5,5,7,10)
11 I

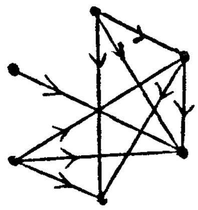
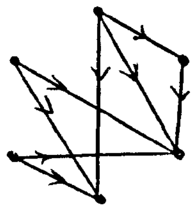
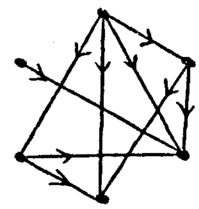
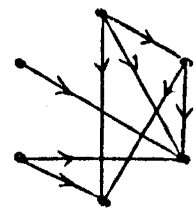
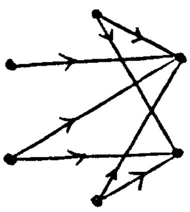
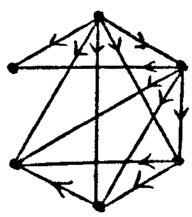
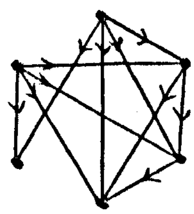
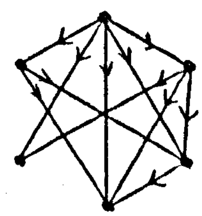
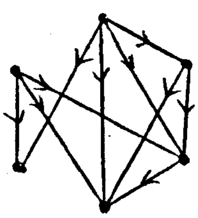
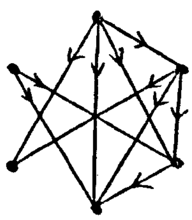
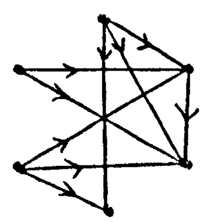
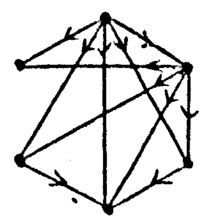
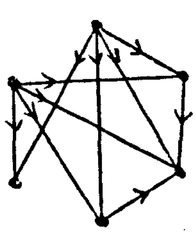
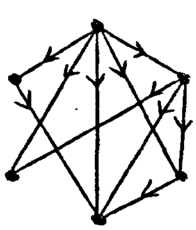
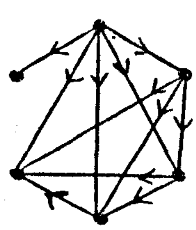
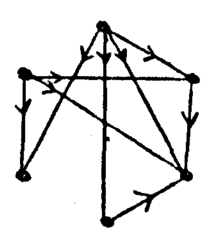
92 720

(1,2,5,5,8,9)
10 I

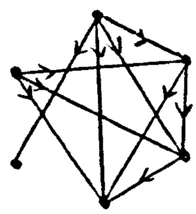
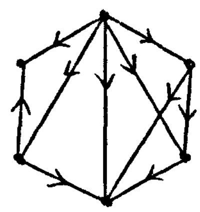
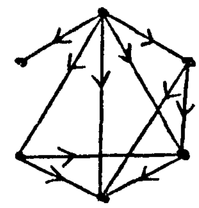
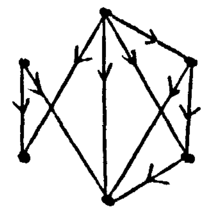
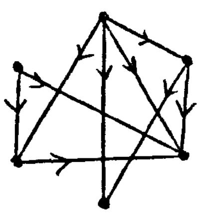
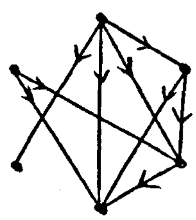
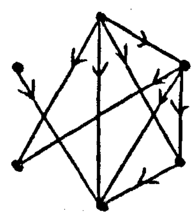
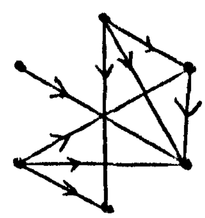
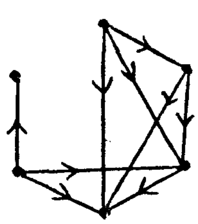
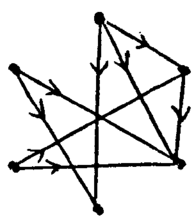
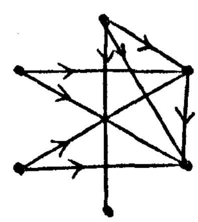
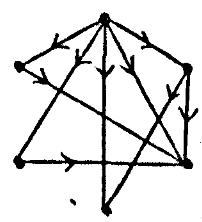
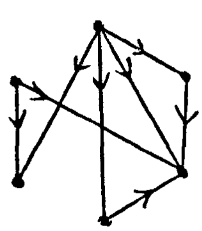
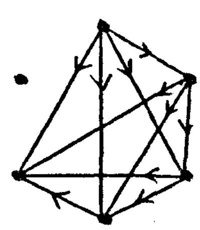
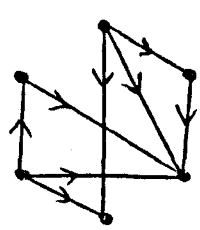
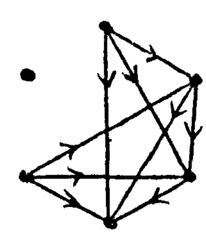
93 360

(1,2,5,6,6,10)
10 S S_2

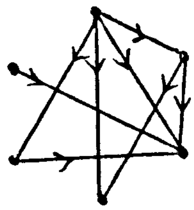
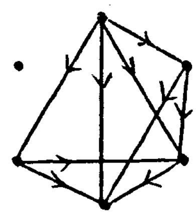
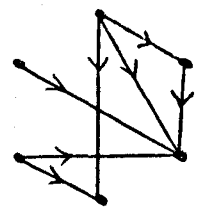
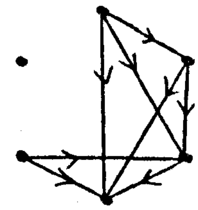
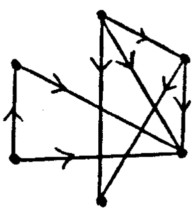
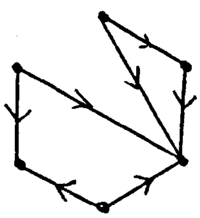
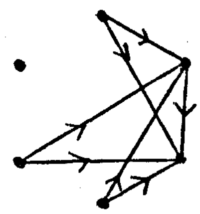
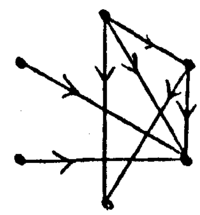
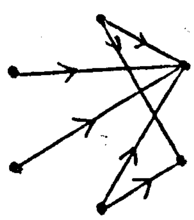
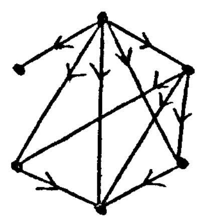
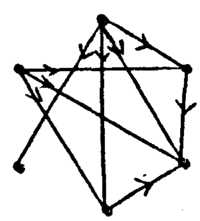
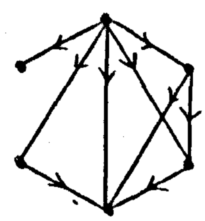
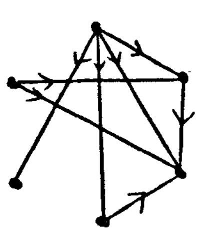
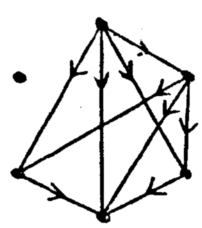
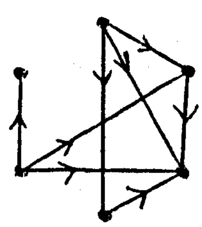
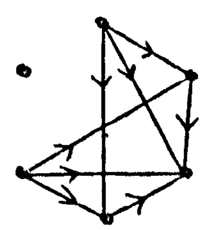
94 720

(1,2,5,6,7,9)
9 S I

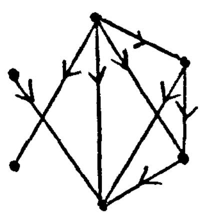
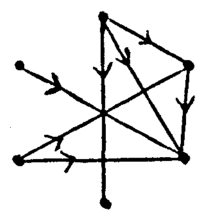
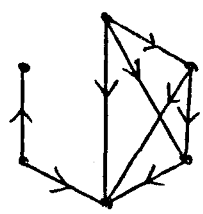
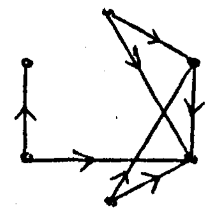
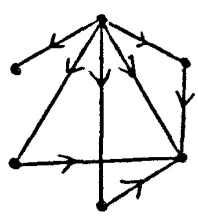
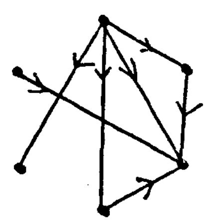
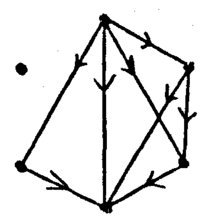
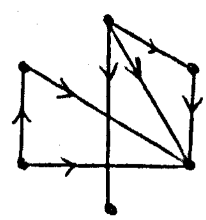
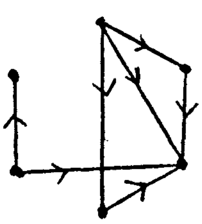
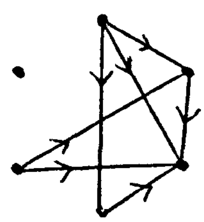
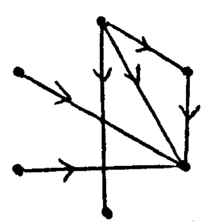
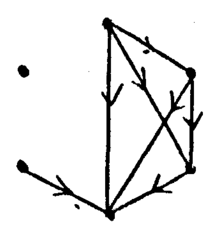
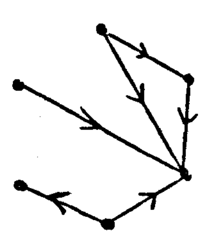
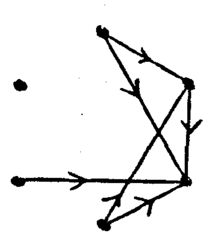
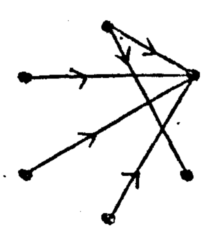
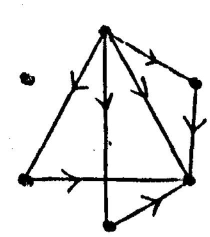
95 720

(1,2,5,6,7,9)
10 I

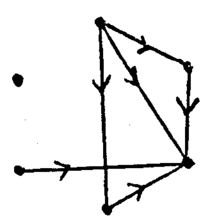
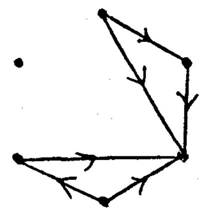
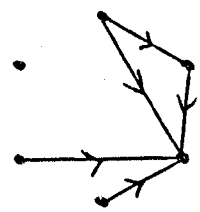
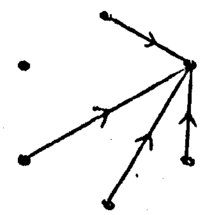
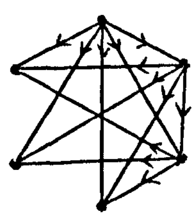
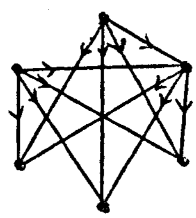
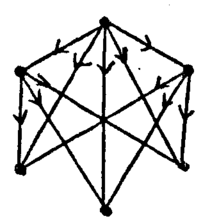
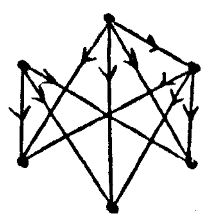
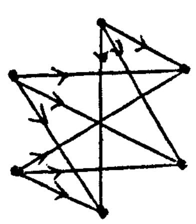
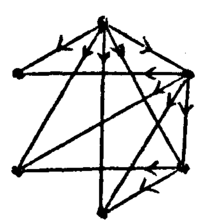
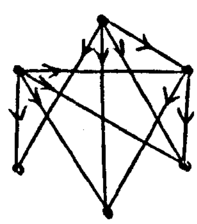
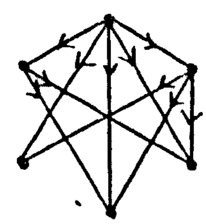
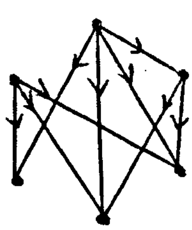
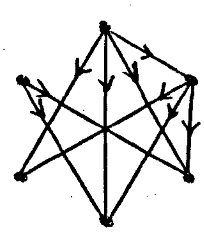
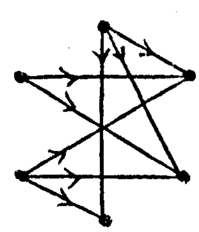
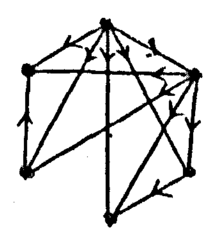
96 720

(1,2,5,6,8,8)
9 I

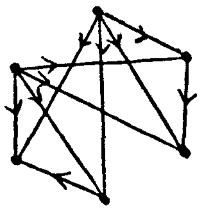
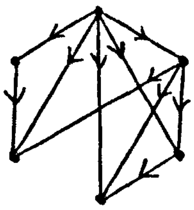
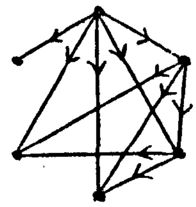
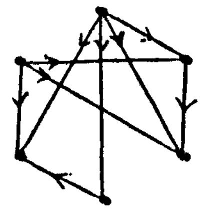
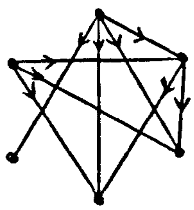
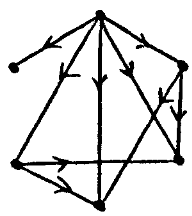
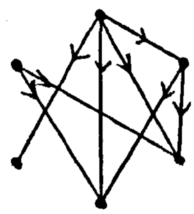
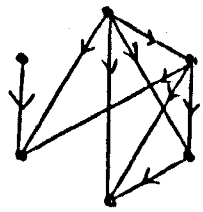
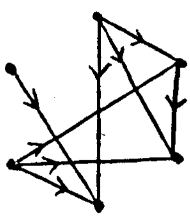
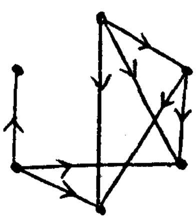
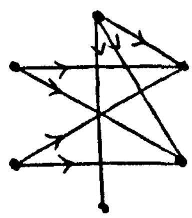
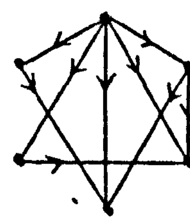
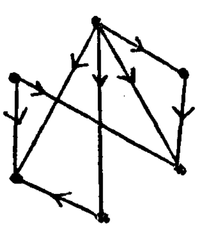
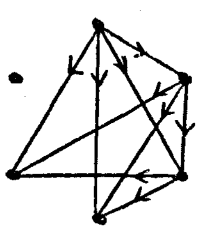
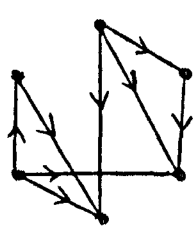
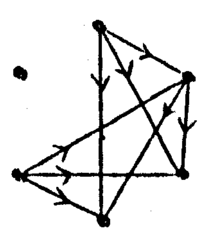
97 360  (1,2,5,6,8,8) 9 S_2	98 360  (1,2,5,7,7,8) 8 S S_2	99 360  (1,2,6,6,6,9) 9 S_2	100 720  (1,2,6,6,7,8) 8 I
101 120  (1,2,6,7,7,7) 7 S S_3	102 720  (1,3,3,5,8,10) 12 I	103 360  (1,3,3,5,9,9) 11 S_2	104 720  (1,3,3,6,7,10) 11 I
105 720  (1,3,3,6,8,9) 10 I	106 720  (1,3,3,7,7,9) 10 I	107 360  (1,3,3,7,8,8) 9 S_2	108 360  (1,3,4,4,8,10) 11 S S_2
109 180  (1,3,4,4,9,9) 10 S $S_2 \times S_2$	110 720  (1,3,4,5,7,10) 10 S I	111 720  (1,3,4,5,7,10) 11 I	112 720  (1,3,4,5,8,9) 9 S I

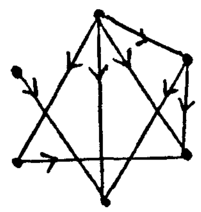
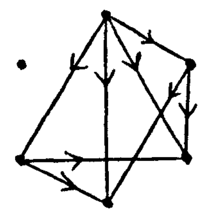
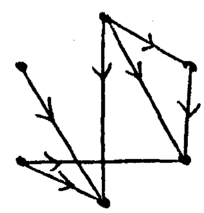
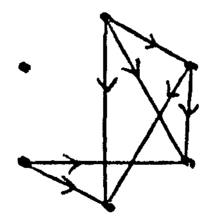
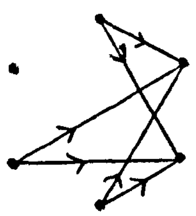
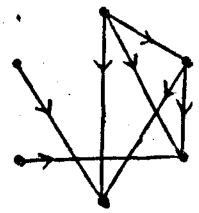
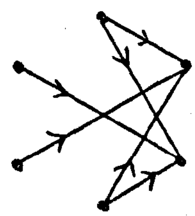
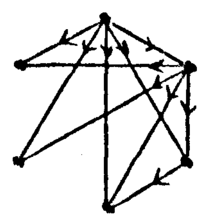
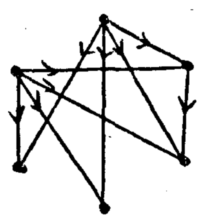
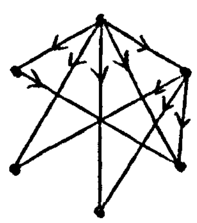
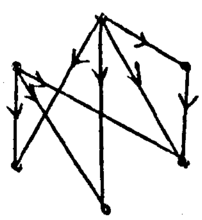
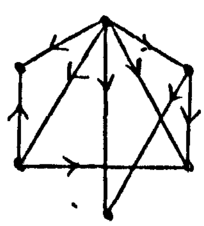
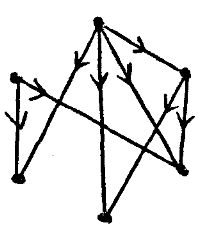
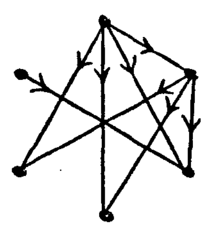
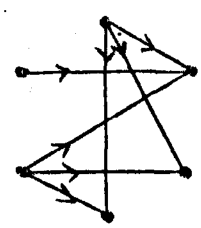
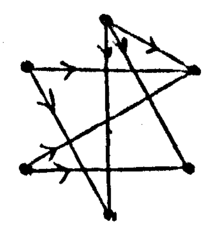
113	720		(1,3,4,5,8,9)	10		I
114	720		(1,3,4,6,6,10)	10		I
115	360		(1,3,4,6,6,10)	10		S_2
116	720		(1,3,4,6,7,9)	9		I
117	720		(1,3,4,6,7,9)	9		I
118	720		(1,3,4,6,7,9)	9		I
119	720		(1,3,4,6,7,9)	9		I
120	360		(1,3,4,6,8,8)	8	S	S_2
121	720		(1,3,4,6,8,8)	9		I
122	720		(1,3,4,7,7,8)	8		I
123	360		(1,3,4,7,7,8)	8		S_2
124	360		(1,3,5,5,6,10)	9	S	S_2
125	360		(1,3,5,5,7,9)	8	S	S_2
126	720		(1,3,5,5,7,9)	10		I
127	360		(1,3,5,5,8,8)	8		S_2
128	360		(1,3,5,5,8,8)	9		S_2

129 720  (1,3,5,6,6,9) 8 I	130 360  (1,3,5,6,6,9) 9 S_2	131 720  (1,3,5,6,7,8) 7 S I	132 720  (1,3,5,6,7,8) 8 I
133 720  (1,3,5,6,7,8) 8 I	134 360  (1,3,5,7,7,7) 7 S_2	135 120  (1,3,5,7,7,7) 7 S_3	136 360  (1,3,6,6,6,8) 7 S_2
137 180  (1,3,6,6,7,7') 6 S $S_2 \times S_2$	138 360  (1,4,4,4,7,10) 10 S_2	139 360  (1,4,4,4,8,9) 9 S_2	140 720  (1,4,4,5,6,10) 9 I
141 720  (1,4,4,5,7,9) 8 I	142 360  (1,4,4,5,7,9) 9 S_2	143 720  (1,4,4,5,8,8) 8 I	144 180  (1,4,4,5,8,8) 8 $S_2 \times S_2$

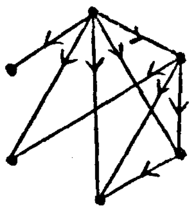
145 720  (1,4,4,6,6,9) 8 I	146 720  (1,4,4,6,7,8) 7 I	147 720  (1,4,4,6,7,8) 8 I	148 360  (1,4,4,7,7,7) 7 S_2
149 120  (1,4,5,5,5,10) 8 S S_3	150 360  (1,4,5,5,6,9) 7 S S_2	151 720  (1,4,5,5,6,9) 8 I	152 720  (1,4,5,5,7,8) 7 I
153 360  (1,4,5,5,7,8) 7 S_2	154 720  (1,4,5,5,7,8) 7 I	155 360  (1,4,5,6,6,8) 6 S S_2	156 720  (1,4,5,6,6,8) 7 I
157 720  (1,4,5,6,7,7) 6 I	158 360  (1,4,5,6,7,7) 6 S_2	159 120  (1,4,6,6,6,7) 5 S S_3	160 120  (1,5,5,5,5,9) 7 S_3

161 360  (1,5,5,5,6,8) 6 S_2	162 360  (1,5,5,5,7,7) 6 S_2	163 360  (1,5,5,6,6,7) 5 S_2	164 30  (1,5,6,6,6,6) 4 S S_4
165 120  (2,2,2,6,8,10) 12 W S_3	166 60  (2,2,2,6,9,9) 11 W $S_2 \times S_3$	167 60  (2,2,2,7,7,10) 11 W $S_2 \times S_3$	168 120  (2,2,2,7,8,9) 10 S S_3
169 20  (2,2,2,8,8,8') 9 W $S_3 \times S_3$	170 360  (2,2,3,5,8,10) 11 S S_2	171 180  (2,2,3,5,9,9) 10 S $S_2 \times S_2$	172 360  (2,2,3,6,7,10) 10 S S_2
173 360  (2,2,3,6,8,9) 9 S S_2	174 360  (2,2,3,7,7,9) 9 S_2	175 180  (2,2,3,7,8,8) 8 S $S_2 \times S_2$	176 360  (2,2,4,4,8,10) 11 S_2

177 180  (2,2,4,4,9,9) 10 $S_2 \times S_2$	178 720  (2,2,4,5,7,10) 10 I	179 360  (2,2,4,5,7,10) 10 S_2	180 720  (2,2,4,5,8,9) 9 I
181 360  (2,2,4,5,8,9) 9 S_2	182 180  (2,2,4,6,6,10) 9 S $S_2 \times S_2$	183 360  (2,2,4,6,7,9) 8 S S_2	184 720  (2,2,4,6,7,9) 9 I
185 360  (2,2,4,6,8,8) 8 S_2	186 360  (2,2,4,6,8,8) 8 S_2	187 180  (2,2,4,7,7,8) 7 S $S_2 \times S_2$	188 360  (2,2,5,5,6,10) 9 S_2
189 360  (2,2,5,5,7,9) 8 S_2	190 360  (2,2,5,5,7,9) 9 S_2	191 360  (2,2,5,5,8,8) 8 S_2	192 180  (2,2,5,5,8,8) 8 $S_2 \times S_2$

193 720  (2,2,5,6,6,9) 8 I	194 180  (2,2,5,6,6,9) 8 $S_2 \times S_2$	195 720  (2,2,5,6,7,8) 7 I	196 360  (2,2,5,6,7,8) 7 S_2
197 60  (2,2,5,7,7,7) 6 S $S_2 \times S_3$	198 360  (2,2,6,6,6,8) 7 S_2	199 180  (2,2,6,6,7,7) 6 $S_2 \times S_2$	200 360  (2,3,3,4,8,10) 10 S S_2
201 180  (2,3,3,4,9,9) 9 S $S_2 \times S_2$	202 360  (2,3,3,5,7,10) 9 S S_2	203 360  (2,3,3,5,8,9) 8 S S_2	204 360  (2,3,3,6,6,10) 9 S_2
205 720  (2,3,3,6,7,9) 8 I	206 360  (2,3,3,6,7,9) 8 S_2	207 180  (2,3,3,6,8,8) 7 S $S_2 \times S_2$	208 360  (2,3,3,7,7,8) 7 S_2

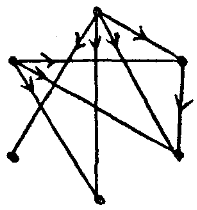
209 720



(2,3,4,4,7,10)

9 I

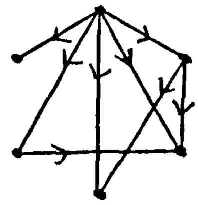
210 720



(2,3,4,4,8,9)

8 I

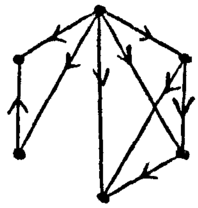
211 720



(2,3,4,5,6,10)

8 S I

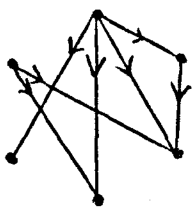
212 720



(2,3,4,5,6,10)

9 I

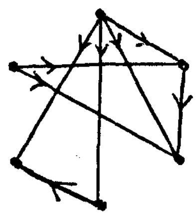
213 720



(2,3,4,5,7,9)

7 S I

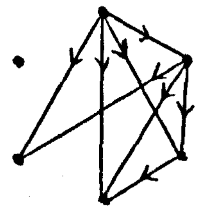
214 720



(2,3,4,5,7,9)

8 I

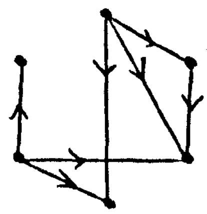
215 720



(2,3,4,5,7,9)

8 I

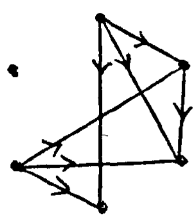
216 720



(2,3,4,5,8,8)

7 I

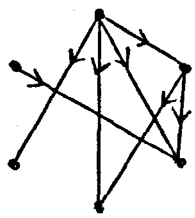
217 360



(2,3,4,5,8,8)

7 S_2

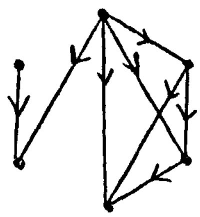
218 720



(2,3,4,6,6,9)

7 I

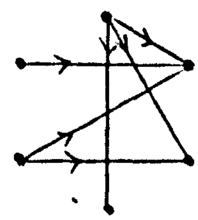
219 720



(2,3,4,6,6,9)

8 I

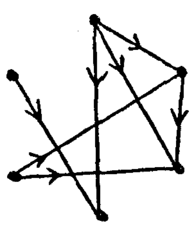
220 720



(2,3,4,6,7,8)

6 S I

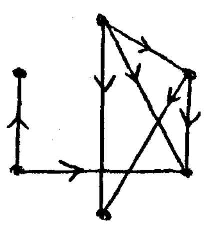
221 720



(2,3,4,6,7,8)

7 I

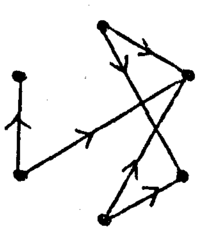
222 720



(2,3,4,6,7,8)

7 I

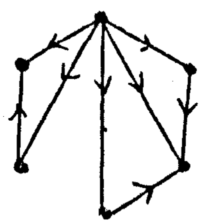
223 360



(2,3,4,7,7,7)

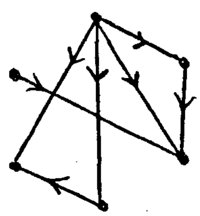
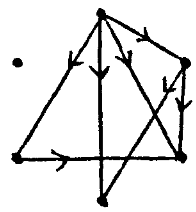
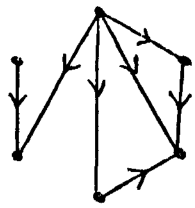
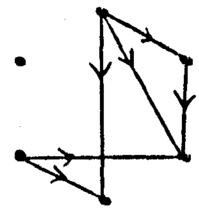
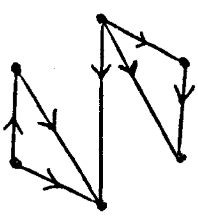
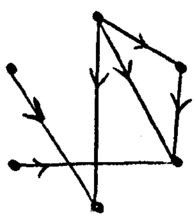
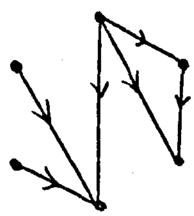
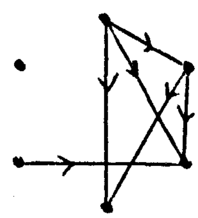
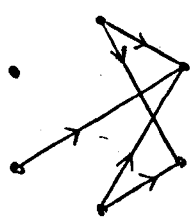
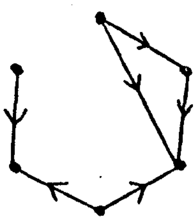
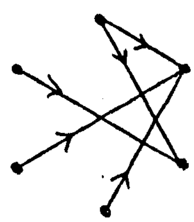
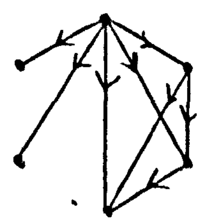
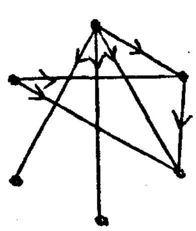
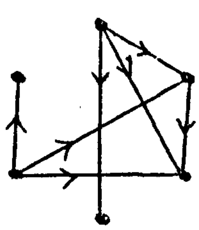
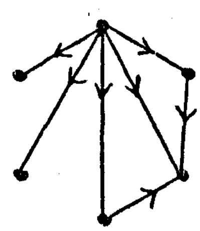
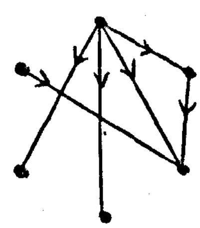
6 S_2

224 360

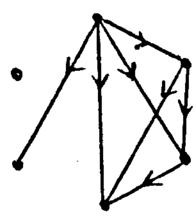


(2,3,5,5,5,10)

8 S_2

225	720		(2,3,5,5,6,9)	7		I
226	720		(2,3,5,5,6,9)	7		I
227	360		(2,3,5,5,6,9)	7		S_2
228	720		(2,3,5,5,7,8)	6		I
229	720		(2,3,5,5,7,8)	7		I
230	720		(2,3,5,6,6,8)	6		I
231	360		(2,3,5,6,6,8)	6		S_2
232	720		(2,3,5,6,6,8)	6		I
233	360		(2,3,5,6,7,7)	5	S	S_2
234	720		(2,3,5,6,7,7)	6		I
235	360		(2,3,6,6,6,7)	5		S_2
236	360		(2,4,4,4,6,10)	8		S_2
237	360		(2,4,4,4,7,9)	7		S_2
238	360		(2,4,4,4,8,8)	7		S_2
239	180		(2,4,4,5,5,10)	7	S	$S_2 \times S_2$
240	360		(2,4,4,5,6,9)	6	S	S_2

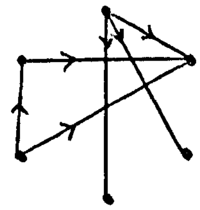
241 720



(2,4,4,5,6,9)

7 I

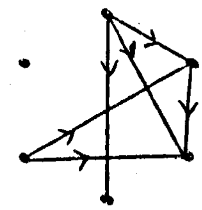
242 360



(2,4,4,5,7,8)

6 S_2

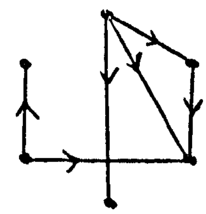
243 720



(2,4,4,5,7,8)

6 I

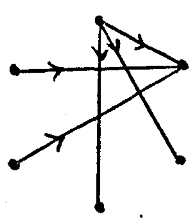
244 720



(2,4,4,5,7,8)

6 I

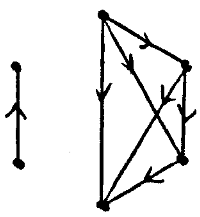
245 180



(2,4,4,6,6,8)

5 S $S_2 \times S_2$

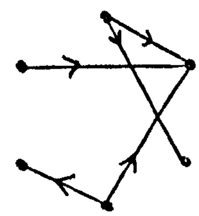
246 720



(2,4,4,6,6,8)

7 I

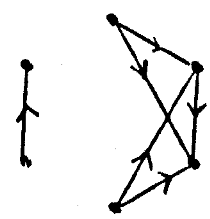
247 360



(2,4,4,6,7,7)

5 S_2

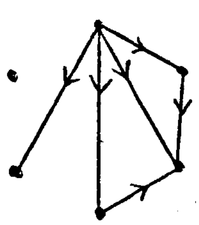
248 360



(2,4,4,6,7,7)

6 S_2

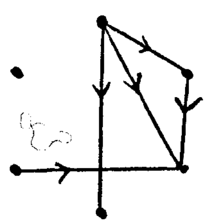
249 360



(2,4,5,5,5,9')

6 S_2

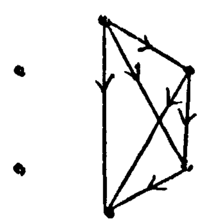
250 720



(2,4,5,5,6,8)

5 I

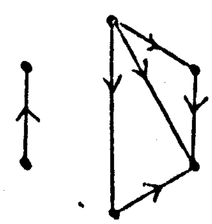
251 360



(2,4,5,5,6,8)

6 S_2

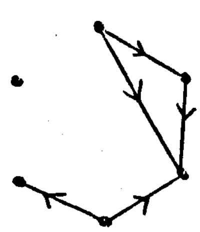
252 360



(2,4,5,5,6,8)

6 S_2

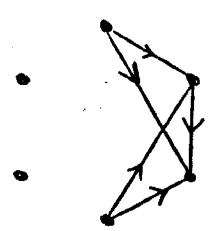
253 720



(2,4,5,5,7,7)

5 I

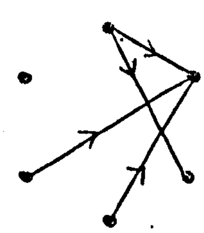
254 180



(2,4,5,5,7,7)

5 $S_2 \times S_2$

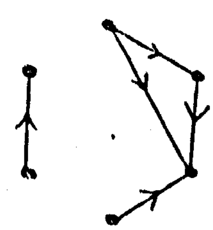
255 360



(2,4,5,6,6,7)

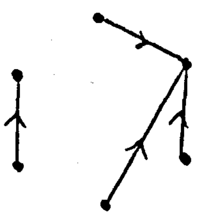
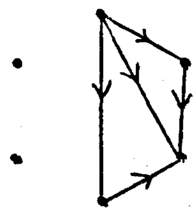
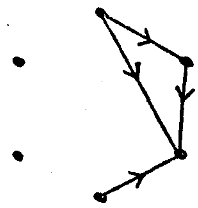
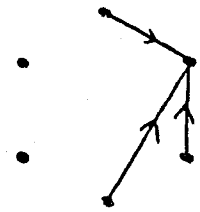
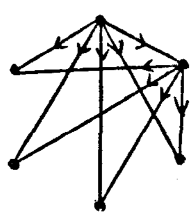
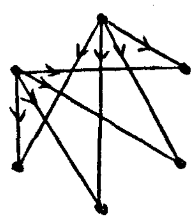
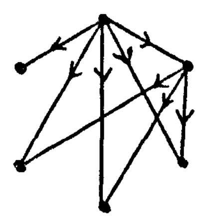
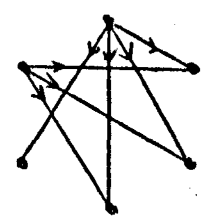
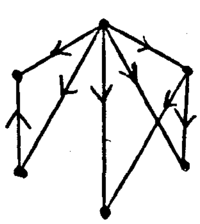
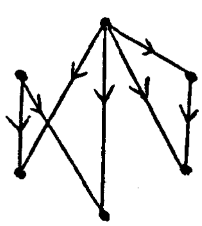
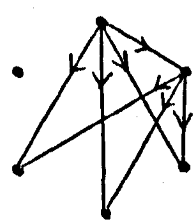
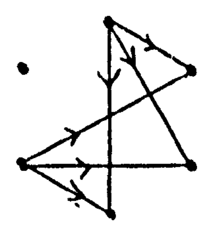
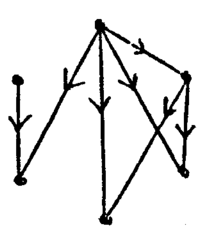
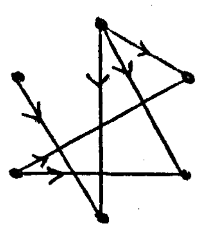
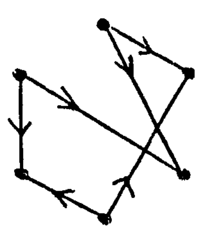
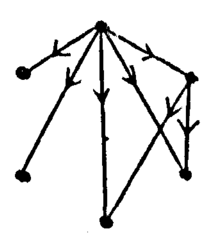
4 S S_2

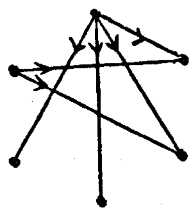
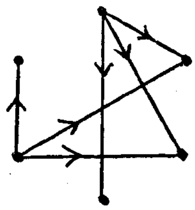
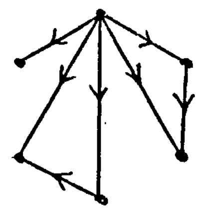
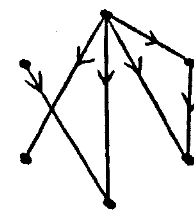
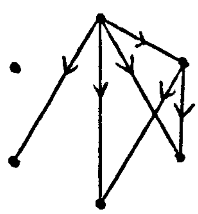
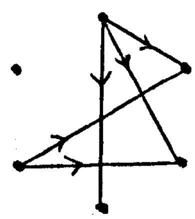
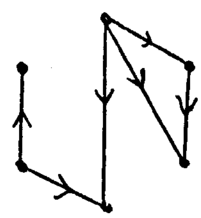
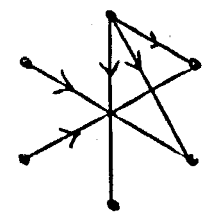
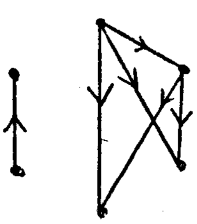
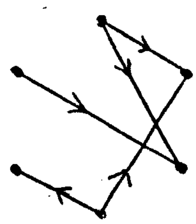
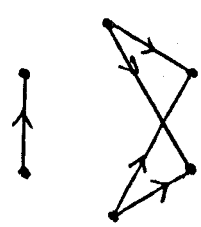
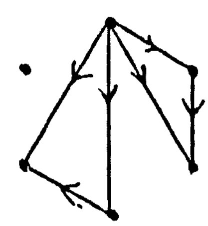
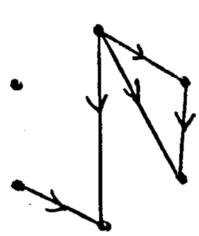
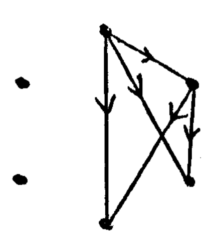
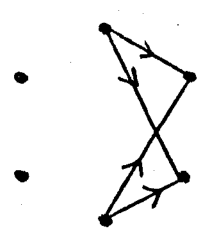
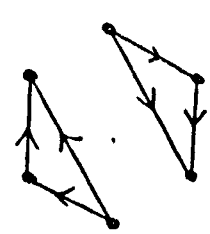
256 720



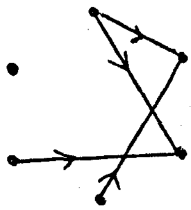
(2,4,5,6,6,7)

5 I

257 120  (2,4,6,6,6,6) 4 S_3	258 180  (2,5,5,5,5,8) 5 $S_2 \times S_2$	259 360  (2,5,5,5,6,7) 4 S_2	260  (2,5,5,6,6,6) 3 S $S_2 \times S_3$
261 30  (3,3,3,3,8,10) 9 W S_4	262 15  (3,3,3,3,9,9) 8 W $S_2 \times S_4$	263 120  (3,3,3,4,7,10) 8 S S_3	264 120  (3,3,3,4,8,9) 7 S S_3
265 360  (3,3,3,5,6,10) 8 S_2	266 360  (3,3,3,5,7,9) 7 S_2	267 120  (3,3,3,5,7,9) 7 S_3	268 60  (3,3,3,5,8,8) 6 S $S_2 \times S_3$
269 360  (3,3,3,6,6,9) 7 S_2	270 360  (3,3,3,6,7,8) 6 S_2	271 120  (3,3,3,7,7,7) 6 S_3	272 180  (3,3,4,4,6,10) 7 S $S_2 \times S_2$

273 180	274 180	275 360	276 720
			
(3,3,4,4,7,9)	(3,3,4,4,8,8)	(3,3,4,5,5,10)	(3,3,4,5,6,9)
6 S $S_2 \times S_2$	6 $S_2 \times S_2$	7 S_2	6 I
277 360	278 360	279 720	280 360
			
(3,3,4,5,6,9)	(3,3,4,5,7,8)	(3,3,4,5,7,8)	(3,3,4,6,6,8)
6 S_2	5 S S_2	6 I	5 S_2
281 360	282 720	283 180	284 360
			
(3,3,4,6,6,8')	(3,3,4,6,7,7)	(3,3,4,6,7,7)	(3,3,5,5,5,9)
6 S_2	5 I	5 $S_2 \times S_2$	6 S_2
285 720	286 180	287 90	288 360
			
(3,3,5,5,6,8)	(3,3,5,5,6,8)	(3,3,5,5,7,7)	(3,3,5,5,7,7)
5 I	5 $S_2 \times S_2$	4 S $S_2 \times S_2 \times S_2$	6 S_2

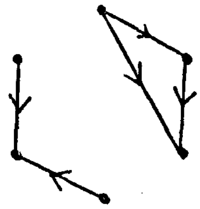
289 360



(3,3,5,6,6,7)

4 S_2

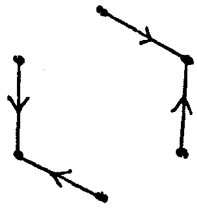
290 360



(3,3,5,6,6,7)

5 S_2

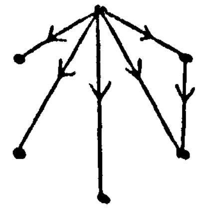
291 90



(3,3,6,6,6,6)

4 D_8

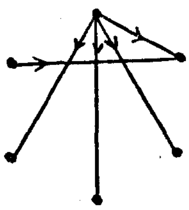
292 120



(3,4,4,4,5,10)

6 S S_3

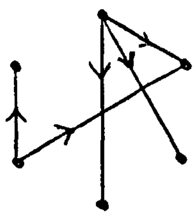
293 120



(3,4,4,4,6,9)

5 S S_3

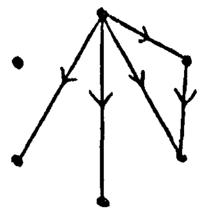
294 360



(3,4,4,4,7,8)

5 S_2

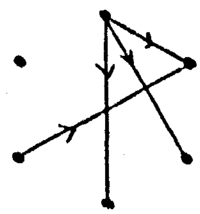
295 360



(3,4,4,5,5,9)

5 S_2

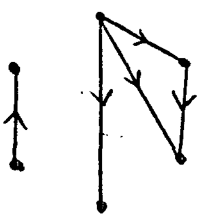
296 360



(3,4,4,5,6,8)

4 S S_2

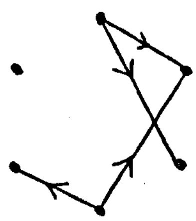
297 720



(3,4,4,5,6,8')

5 I

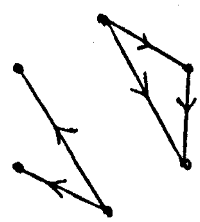
298 360



(3,4,4,5,7,7)

4 S_2

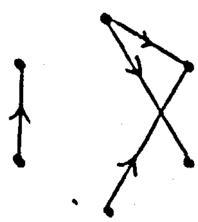
299 360



(3,4,4,5,7,7)

5 S_2

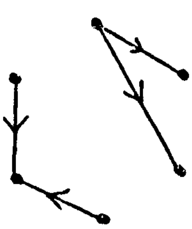
300 720



(3,4,4,6,6,7)

4 I

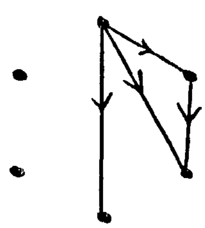
301 180



(3,4,4,6,6,7)

4 $S_2 \times S_2$

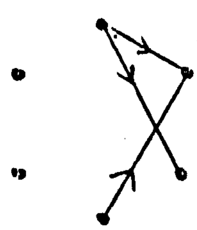
302 360



(3,4,5,5,5,8)

4 S_2

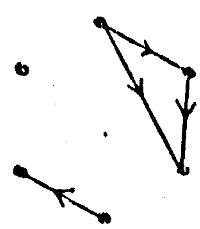
303 360



(3,4,5,5,6,7)

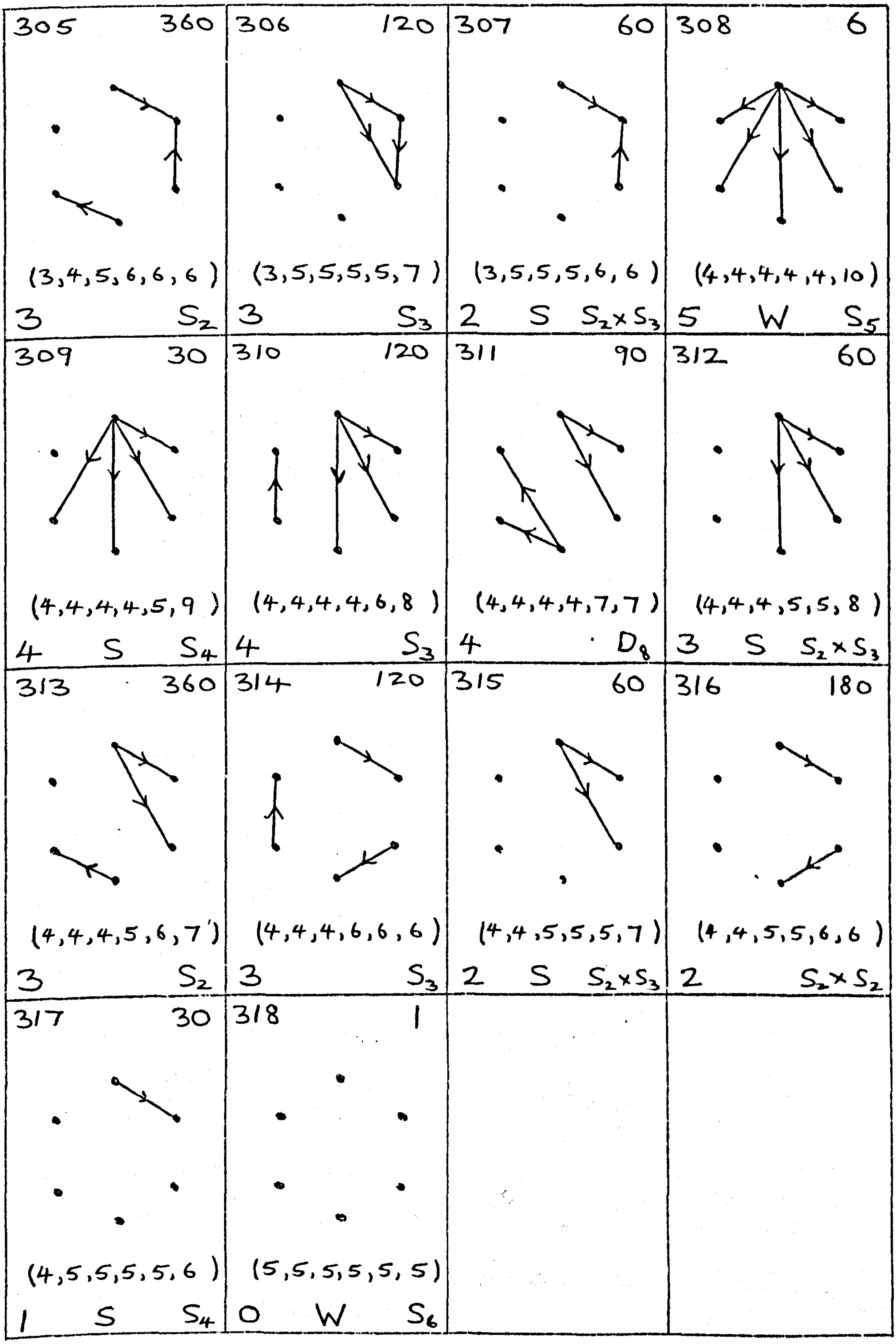
3 S S_2

304 720



(3,4,5,5,6,7)

4 I


305 360
(3,4,5,6,6,6)
3 S2
306 120
(3,5,5,5,5,7)
3 S3
307 60
(3,5,5,5,6,6)
2 S S2×S3
308 6
(4,4,4,4,4,10)
5 W S5
309 30
(4,4,4,4,5,9)
4 S S4
310 120
(4,4,4,4,6,8)
4 S3
311 90
(4,4,4,4,7,7)
4 D8
312 60
(4,4,4,5,5,8)
3 S S2×S3
313 360
(4,4,4,5,6,7')
3 S2
314 120
(4,4,4,6,6,6)
3 S3
315 60
(4,4,5,5,5,7)
2 S S2×S3
316 180
(4,4,5,5,6,6)
2 S2×S2
317 30
(4,5,5,5,5,6)
1 S S4
318 1
(5,5,5,5,5,5)
0 W S6

R E F E R E N C E S

1. R. Alter, "Some remarks and results on Catalan numbers", in *Proc. second Louisiana conf. on combinatorics, graph theory and computing* (eds. Mullin, Reid, Roselle and Thomas), Louisiana State Univ. (Baton Rouge, La.), 1971, 109-132.

2. W.E. Armstrong, "A note on the theory of consumer's behaviour", *Oxford Economic Papers* 2 (1950), 119-122.

3. W.E. Armstrong, "Utility and the theory of welfare", *Oxford Economic Papers* 3 (1951), 259-271.

4. D.W. Bass, "Graphical representation of finite topological spaces", *Ann. Soc. Sci. Bruxelles Sér. I* 86 (1972), 193-200.

5. C. Berge, "*Graphs and hypergraphs*", North-Holland (Amsterdam), 1973.

6. G. Birkhoff, "*Lattice theory*", Amer. Math. Soc. Colloq. Publ. Vol. 25, Revised Edition (Providence, R.I.), 1948.

7. D. Blackwell and M.A. Girshick, "*Theory of games and statistical decisions*", Wiley (New York), 1954.

8. S.K. Das, "On the enumeration of finite maximal connected topologies", *J. Combinatorial Theory, Ser. B* 15(1973), 184-199.

9. H.A. David, "*The method of paired comparisons*", Griffin (London), 1963.

10. R.L. Davis, "The number of structures of finite relations", *Proc. Amer. Math. Soc.* 4(1953), 486-495.

11. D.B. Duncan, "Multiple range and multiple F tests", *Biometrics* 11 (1955), 1-42.

12. J.W. Evans, F. Harary and M.S. Lynn, "On the computer enumeration of finite topologies", *Comm. Assoc. Comp. Mach.* 10(1967), 295-298.

13. W.T. Federer, "*Experimental design*", Macmillan (New York), 1955.

14. S. Hakimi, "On the realizability of a set of integers as degrees of the vertices of a graph", *J. SIAM Appl. Math.* 10(1962), 496-506.

15. M. Hall, "*Combinatorial theory*", Blaisdell (Waltham), 1967.

16. F. Harary, "On the reconstruction of a graph from a collection of subgraphs", in *Theory of graphs and its applications* (ed. Fiedler), Prague, 1964, 47-52; reprinted Academic Press (New York), 1964.

17. F. Harary, R.Z. Norman and D. Cartwright, "*Structural models: an introduction to the theory of directed graphs*", Wiley (New York), 1965.

18. F. Harary, "*Graph theory*", Addison-Wesley (Reading, Massachusetts), 1969.

19. F. Harary and E.M. Palmer, "*Graphical enumeration*", Academic Press (New York), 1973.

20. V. Havel, "A remark on the existence of finite graphs", *Časopis Pěst. Mat.* 80(1955), 477-480.

21. M.G. Kendall and B. Babington Smith, "On the method of paired comparisons", *Biometrika* 31(1940), 324-345.

22. M.G. Kendall, "Further contributions to the theory of paired comparisons", *Biometrics* 11(1955), 43-62.

23. A. Kotzig, "On δ-transformations of antisymmetric graphs", *Mat.-Fyz. Časopis Sloven. Akad. Vied* 16(1966), 353-356.

24. V. Krishnamurthy, "On the number of topologies on a finite set", *Amer. Math. Monthly* 73(1966), 154-157.

25. V. Krishnamurthy, "Counting of finite topologies and a dissection of Stirling numbers of the second kind", *Bull. Austral. Math. Soc.* 12(1975), 111-124.

26. K. Kuratowski, "*Topology*", Volume 1, Academic Press(New York), 1966.

27. H.G. Landau, "On dominance relations and the structure of animal societies: I. Effect of inherent characteristics", *Bull. Math. Biophysics* 13(1951), 1-19.

28. H.G. Landau, "On dominance relations and the structure of animal societies: II. Some effects of possible social factors", *Bull. Math. Biophysics* 13(1951), 245-262.

29. H.G. Landau, "On dominance relations and the structure of animal societies: III. The condition for a score structure", *Bull. Math. Biophysics* 15(1953), 143-148.

30. R.D. Luce, "Semiorders and a theory of utility discrimination", *Econometrica* 24(1956), 178-191.

31. B. Manvel, "On reconstruction of graphs", in *The many facets of graph theory* (eds. Chartrand and Kapoor), Springer-Verlag (New York), 1969, 207-214.

32. B. Manvel, "Reconstructing the degree pair sequence of a digraph", *J. Combinatorial Theory, Ser. B* 15(1973), 18-31.

33. J.W. Moon, *"Topics on tournaments"*, Holt, Rinehart and Winston (New York), 1968.

34. T.V. Narayana and D.H. Bent, "Computation of the number of score sequences in round-robin tournaments", *Canad. Math. Bull.* 7(1964), 133-136.

35. R. Otter, "The number of trees", *Ann. of Math.* 49(1948), 583-599.

36. R. Parchmann, "On the cardinalities of finite topologies", *Discrete Mathematics* 11(1975), 161-172.

37. J. Riordan, "The number of score sequences in tournaments", *J. Combinatorial Theory* 5(1968), 87-89.

38. F.S. Roberts, "Indifference graphs", in *Proof techniques in graph theory* (ed. Harary), Academic Press (New York), 1969, 139-146.

39. H.J. Ryser, "Matrices of zeros and ones in combinatorial mathematics", in *Recent advances in matrix theory* (ed. Schneider), University of Wisconsin Press, 1964 , 103-124.

40. G. Sabidussi, "Graph multiplication", *Math. Z.* 72(1960), 446-457.

41. L.J. Savage, "*The foundations of statistics*", Wiley (New York), 1954.

42. D. Scott and P. Suppes, "Foundational aspects of theories of measurement", *J. Symbolic Logic* 23(1958), 113-128.

43. H. Sharp, "Quasi-orderings and topologies on finite sets", *Proc. Amer. Math. Soc.* 17(1966), 1344-1349.

44. H. Sharp, "Cardinality of finite topologies", *J. Combinatorial Theory* 5(1968), 82-86.

45. P. Slater, "Inconsistencies in a schedule of paired comparisons", *Biometrika* 48(1961), 303-312.

46. R.P. Stanley, "On the number of open sets of finite topologies", *J. Combinatorial Theory* 10(1971), 74-79.

47. D. Stephen, "Topology on finite sets", *Amer. Math. Monthly* 75(1968), 739-741.

48. J.P. Thomas, "Maximal connected topologies", *J. Austal. Math. Soc.* 8(1968), 700-705.

49. S.M. Ulam, "*A collection of mathematical problems*", Wiley (New York), 1960.

50. R.L. Wine and J.E. Freund, "On the enumeration of decision patterns involving n means", *Ann. Math. Stat.* 28(1957), 256-259.